Step 2 (a) By $\pi_1 : N := M \times [0,1] \longrightarrow M$

and $\pi_2 : N \longrightarrow \cancel{M}\ [0,1]$

$(p,s) \longmapsto s$

We obtain $T_{(p,s)}\pi_1 : T_{(p,s)}N \twoheadrightarrow T_p M$

and $T_{(p,s)}\pi_2 : T_{(p,s)}N \twoheadrightarrow T_s[0,1]$

and $\underbrace{(T_{(p,s)}\pi_1) \oplus T_{(p,s)}\pi_2}_{\Phi_{(p,s)}} : T_{(p,s)}N \xrightarrow{\sim} T_p M \oplus T_s[0,1]$

Take $v \in T_p M$. We define the vector field $X_v : [0,1] \longrightarrow TN$ via $X_v(s) := \Phi^{-1}_{(p,s)}(v, 0)$

(b) Define $\frac{\partial}{\partial t}(p,s) := \Phi^{-1}_{(p,s)}\left(0, \frac{\partial}{\partial t}(s)\right)$

This is the tangent vector for the curve $(-\varepsilon, \varepsilon) \xrightarrow{c} N$, $c(t) := (p, s+t)$, see Prop. 1.26(a).

(c) We define L: $\omega \in \Omega^k(N)$.

<u>$k=0$:</u> $L(\omega) := 0$.

<u>$k>0$:</u> $L(\omega)_p(v_1, \dots, v_{k-1})$

$$:= \int_0^1 \omega_{(p,s)}\left(\frac{\partial}{\partial t}(p,s), X_{v_1}(s), \dots, X_{v_{k-1}}(s)\right) ds.$$

Step 3: By definition of L,
L is local on M, i.e. if $\exists\, U$ open $\subseteq M$ and $\omega_1, \omega_2 \in \Omega^*(N)$ s.t. $\omega_1|_{U\times[0,1]} = \omega_2|_{U\times[0,1]}$, then

$$L(\omega_1)|_U = L(\omega_2)|_U.$$

Thus by additivity of L we only need to consider

$$\omega = a\,dx^I \quad \text{or} \quad \omega = a\, dt \wedge dx^I$$

for a chart (U, φ) of M and $a \in C^\infty_{c_\pi}(U \times [0,1], \mathbb{R})$.
compact support.

Step 4: $d_M L + L d_N = f_1^* - f_0^*$.

(a) $\omega = a \in C_c^\infty(U \times [0,1])$.

- $(f_1^*(a) - f_0^*(a))(p) = (a\circ f_1)(p) - (a \circ f_0)(p)$
  $= a(p,1) - a(p,0)$

$(d_M L)(a) = d_M(L(a)) = d_M 0 = 0.$ (zero form)

$(L d_N)(a)(P) = L\left(\sum_{i=1}^{m} \frac{\partial a}{\partial x_i} dx_i + \frac{\partial a}{\partial t} dt\right)(P)$

$\underset{\underset{dx_i\left(\frac{\partial}{\partial t}\right)=0}{\uparrow}}{=} \int_0^1 \frac{\partial a}{\partial t}(P,s)\, ds$

$= a(P,1) - a(P,0)$

This proves the case (a).

(b) $\omega = a\, dx^I$, $a \in C_c^\infty(U \times [0,1])$

$1 \leq \underset{=:k}{|I|} \leq m.$

- $L(a\,dx^I) = 0$, because $(dx_j)_{(P,s)}\left(\frac{\partial}{\partial t}(P,s)\right) = 0 \quad \forall_{1 \leq j \leq m}.$
- $(L\, d_N)(a\, dx^I) = L\left(\frac{\partial a}{\partial t} dt \wedge dx^I\right)$

we plug in $v_1, \dots, v_{k-1} \in T_P M.$

$L\left(\frac{\partial a}{\partial t} dt \wedge dx^I\right)_P (v_1, \dots, v_{k-1})$

$= \int_0^1 \frac{\partial a}{\partial t}(P,s)\, (dx^I)_{(P,s)}\left(X_{v_1}^{(s)}, \dots, X_{v_{k-1}}^{(s)}\right) ds.$

$$\overset{\uparrow}{=} \left(\int_0^1 \frac{\partial a}{\partial t}(p,s)\, ds\right)(dx^I)_p(v_1,\dots,v_{k-1})$$

$$\left[(dx^I)_{(p,s)}(X_{v_1}(s),\dots,X_{v_{k-1}}(s)) = (dx^I)_p(v_1,\dots,v_{k-1})\right]$$

$$= (f_1^*(a)(p) - f_0^*(a)(p))\,(dx^I)_p(v_1,\dots,v_{k-1})$$

$$= (f_1^*(a\,dx^I) - f_0^*(a\,dx^I))_p(v_1,\dots,v_{k-1}).$$

(c) $\omega = a\,dt \wedge dx^I$; $a \in C_c^\infty(U\times[0,1],\mathbb{R})$

- $f_1^*(\omega) - f_0^*(\omega) = \underbrace{f_1^*(a\,dt)}_{=0} \wedge f_1^*(dx^I) - \underbrace{f_0^*(a\,dt)}_{=0} \wedge f_0^*(dx^I)$

$$= 0 - 0 = 0$$

- $(d_M L)(\omega)_p = d_M\left(\int_0^1 a(\cdot,s)\,ds\,(dx^I)\right)(p)$

$$= \int_0^1 d_M(a(-,s))_p\,ds \wedge (dx^I)_p$$

$$+ \int_0^1 a(p,s)\,ds\,(\underbrace{d_M\,dx^I}_{=0})_p$$

- $(L\,d_N)(\omega)_p = \cancel{L(\ldots)_p}$

$$= L(da \wedge dt \wedge dx^I)_p$$

$$= -\int_0^1 \sum_{j=1}^m \frac{\partial a}{\partial x_j}(p,s)\,ds \cdot dx_j \wedge dx^I$$

$$= -\int_0^1 d_M(a(-,s))_p\,ds \wedge dx^I \qquad \square$$

Theorem 3.61: Suppose $f, g \in C^\infty(M,N)$ are (smoothly) homotopic, i.e. $\exists\ H \in C^\infty(M \times [0,1], N)$ such that $H(-,0) = f$ and $H(-,1) = g$.

Then f^* and g^* agree on $H^*_{dR}(N)$.

Proof: We have $f_i : M \longrightarrow M \times [0,1]$, $i = 0,1$.

Take L from lemma 3.59.

Then
$$H \circ f_1 = g$$
$$H \circ f_0 = f$$

$$\Rightarrow g^* - f^* = (f_1^* - f_0^*) \circ H^*$$
$$= d_M L H^* + L d_{M \times [0,1]} H^*$$
$$\underset{3.56}{=} d_M L H^* + L H^* d_N \qquad \square$$

Remark 3.62: By approximation theory, see Chapter II, we can replace a continuous homotopy H by a smooth one, if $\partial M = \emptyset = \partial N$. We therefore have the following corollary.

Corollary 3.63. Suppose M and N are homotopic mf. without boundary.

Then $H^*_{dR}(N) \simeq H^*_{dR}(M)$.

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Proof: M and N are homotopic topological spaces, i.e. by definition $\exists f: M \to N$ $g: N \to M$ continuous such that $g \circ f$ is homotopic to id_M and $f \circ g$ —"— id_N.

By Remark 3.62 and Theorem 3.61.
$f^* \circ g^* = (g \circ f)^*$ and $(id_M)^*$ agree on de Rham cohomology and
$g^* \circ f^* = (f \circ g)^*$ and $(id_N)^*$ agree on de Rham cohomology.

Now $(id_M)^*|_{H^k_{dR}(M)} = id_{H^k_{dR}(M)}$

and $(id_N)^*|_{H^k_{dR}(N)} = id_{H^k_{dR}(N)}$.

$\square$

Remark 3.64: We can allow boundary if we assume smoothly homotopic in Corollary 3.63.

Example 3.65: $n \in \mathbb{N}$

S^n and $\mathbb{R}^{n+1} \setminus \{\underline{0}\}$ are homotopic:

$$S^n \xhookrightarrow{\iota} \mathbb{R}^{n+1} \setminus \{\underline{0}\} \xrightarrow{\pi} S^n$$
$$\underline{x} \longmapsto \frac{1}{|\underline{x}|_2}\underline{x}.$$

$\pi \circ \iota = id_{S^n}$.

$\iota \circ \pi$ is homotopic to $id_{\mathbb{R}^{n+1} \setminus \{\underline{0}\}}$ via

$$H : (\mathbb{R}^{n+1} \setminus \{\underline{0}\}) \times [0,1] \longrightarrow \mathbb{R}^{n+1} \setminus \{\underline{0}\}$$

$$H(\underline{x}, t) := t\underline{x} + (1-t)\frac{1}{|\underline{x}|_2}\underline{x}.$$

$$H(\underline{x}, 1) = id_{\mathbb{R}^{n+1} \setminus \{\underline{0}\}}(\underline{x})$$
$$H(\underline{x}, 0) = \iota \circ \pi(\underline{x}).$$

Thus $H^*_{dR}(S^n) \cong H^*(\mathbb{R}^{n+1} \setminus \{0\})$

(b) Suppose M and N are mf and N is smoothly homotopic to a point.

Then $H^*_{dR}(M) \simeq H^*_{dR}(M \times N)$
$= H^*_{dR}(M \times N)$

(Exercise !)

<u>Definition 3.66:</u> (cup product)

Let M be a mf.. $H^*_{dR}(M) = \bigoplus_{i=0}^{\infty} H^i_{dR}(M)$

is a graded $\mathbb{R}$-algebra w.r.t. the following product:

$$\cup : H^*_{dR}(M) \times H^*_{dR}(M) \to H^*_{dR}(M)$$

defined on homogenous classes via

$$[\omega_1] \cup [\omega_2] := [\omega_1 \wedge \omega_2]$$

for $[\omega_1] \in H^k_{dR}(M)$ and $[\omega_2] \in H^l_{dR}(M)$.
The map $\cup$ is called the <u>cup product</u>

Prop 3.67: The cup product for M is well-defined.

Proof: Take $[\omega_1] \in H^k_{dR}(M)$ and $[\omega_2] \in H^l_{dR}(M)$ and $\rho \in \Omega^{k-1}(M)$.

Then $[\omega_1 + d\rho] = [\omega_1]$
We have to show
$[(\omega_1 + d\rho) \wedge \omega_2] = [\omega_1 \wedge \omega_2]$, i.e.
we have to show that $d\rho \wedge \omega_2$ is exact.

$$d\rho \wedge \omega_2 = d\rho \wedge \omega_2 + (-1)^{k-1} \rho \wedge \underbrace{d\omega_2}_{=0}$$

$$= d(\rho \wedge \omega_2)$$

is exact. $\square$

Example 3.68: $M = S^1$

$H^*_{dR}(M) \simeq \mathbb{R} \oplus \mathbb{R}$ as $\mathbb{R}$-v.s.

and

$(H^*_{dR}(M), \cup) \simeq \mathbb{R}[X] / (X^2)$ as $\mathbb{R}$-algebras.

$[d\theta] \longmapsto X$

We need better tools to compute $H^n_{dR}(S^n)$ and $H^0_{dR}(\mathbb{R}P^n)$ and $H^*_{dR}(S^1 \times$

<u>Theorem 3.69:</u> Let M be a manifold and $U, V \subseteq M$ open subset such that $U \cup V = M$.

(1) Then the sequence of $\mathbb{R}$-v.s.

$$0 \longrightarrow \Omega^*(U \cup V) \xrightarrow{\alpha} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{\beta} \Omega^*(U \cap V) \rightarrow 0$$

with $\alpha = (\mathrm{incl}_U)^* + (\mathrm{incl}_V)^*$

and $\beta = (\mathrm{incl}_{U \cap V, U})^* - (\mathrm{incl}_{U \cap V, V})^*$

(the summands are just restrictions) is exact, i.e.

- α is injective
- β is surjective and
- $\mathrm{im}(\alpha) = \ker(\beta)$.

(2) We have a long exact sequence

$$0 \to H^0_{dR}(U \cup V) \xrightarrow{\alpha} H^0_{dR}(U) \oplus H^0_{dR}(V) \xrightarrow{\beta} H^0_{dR}(U \cap V) \xrightarrow{\sigma}$$

$$\to H^1_{dR}(U \cup V) \xrightarrow{\alpha} H^1_{dR}(U) \oplus H^1_{dR}(V) \xrightarrow{\beta} H^1_{dR}(U \cap V)$$

$$\to H^2_{dR}(U \cup V) \xrightarrow{\alpha} \dots$$

$$\vdots$$

$$\to H^k_{dR}(U \cup V) \xrightarrow{\alpha} H^k_{dR}(U) \oplus H^k_{dR}(V) \xrightarrow{\beta} H^k(U \cap V) \xrightarrow{\sigma}$$

$$\to \dots ,$$

the <u>Mayer - Vietoris sequence</u>.

<u>Proof:</u> (1) · α is injective, because $\deg(\alpha) = 0$ and $\omega \in \Omega^k(U \cup V)$ is determined by $\omega|_U$ and $\omega|_V$.

- $\beta \circ \alpha(\omega) = \omega|_{U \cap V} - \omega|_{U \cap V} = 0$.

$\Rightarrow \operatorname{im}(\alpha) \subseteq \ker(\beta)$.

(2) We have a long exact sequence

$$0 \to H^0_{dR}(U\cup V) \xrightarrow{\alpha} H^0_{dR}(U) \oplus H^0_{dR}(V) \xrightarrow{\beta} H^0_{dR}(U\cap V)$$
$$\xrightarrow{\delta} H^1_{dR}(U\cup V) \xrightarrow{\alpha} H^1_{dR}(U) \oplus H^1_{dR}(V) \xrightarrow{\beta} H^1_{dR}(U\cap V)$$
$$\to H^2_{dR}(U\cup V) \xrightarrow{\alpha} \cdots$$
$$\vdots$$
$$\to H^k_{dR}(U\cup V) \xrightarrow{\alpha} H^k_{dR}(U) \oplus H^k_{dR}(V) \xrightarrow{\beta} H^k(U\cap V)$$
$$\xrightarrow{\delta} \cdots,$$

the <u>Mayer-Vietoris sequence</u>.

<u>Proof:</u> (1) $\cdot$ α is injective, because $\deg(\alpha) = 0$ and $\omega \in \Omega^k(U\cup V)$ is determined by $\omega|_U$ and $\omega|_V$.

$\cdot$ $\beta \circ \alpha(\omega) = \omega|_{U\cap V} - \omega|_{U\cap V} = 0$.

$\Rightarrow \operatorname{im}(\alpha) \subseteq \ker(\beta)$.

Take $\omega_1 \in \Omega^*(U)$ and $\omega_2 \in \Omega^*(V)$
s.t. $\beta((\omega_1, \omega_2)) = 0$.

$\Rightarrow \omega_1|_{U \cap V} = \omega_2|_{U \cap V}$

Define $\omega := \begin{cases} \omega_1 & \text{on } U \\ \omega_2 & \text{on } V \end{cases}$

$\Rightarrow \alpha(\omega) = (\omega_1, \omega_2)$
$\Rightarrow \ker \beta \subseteq \operatorname{im}(\alpha)$

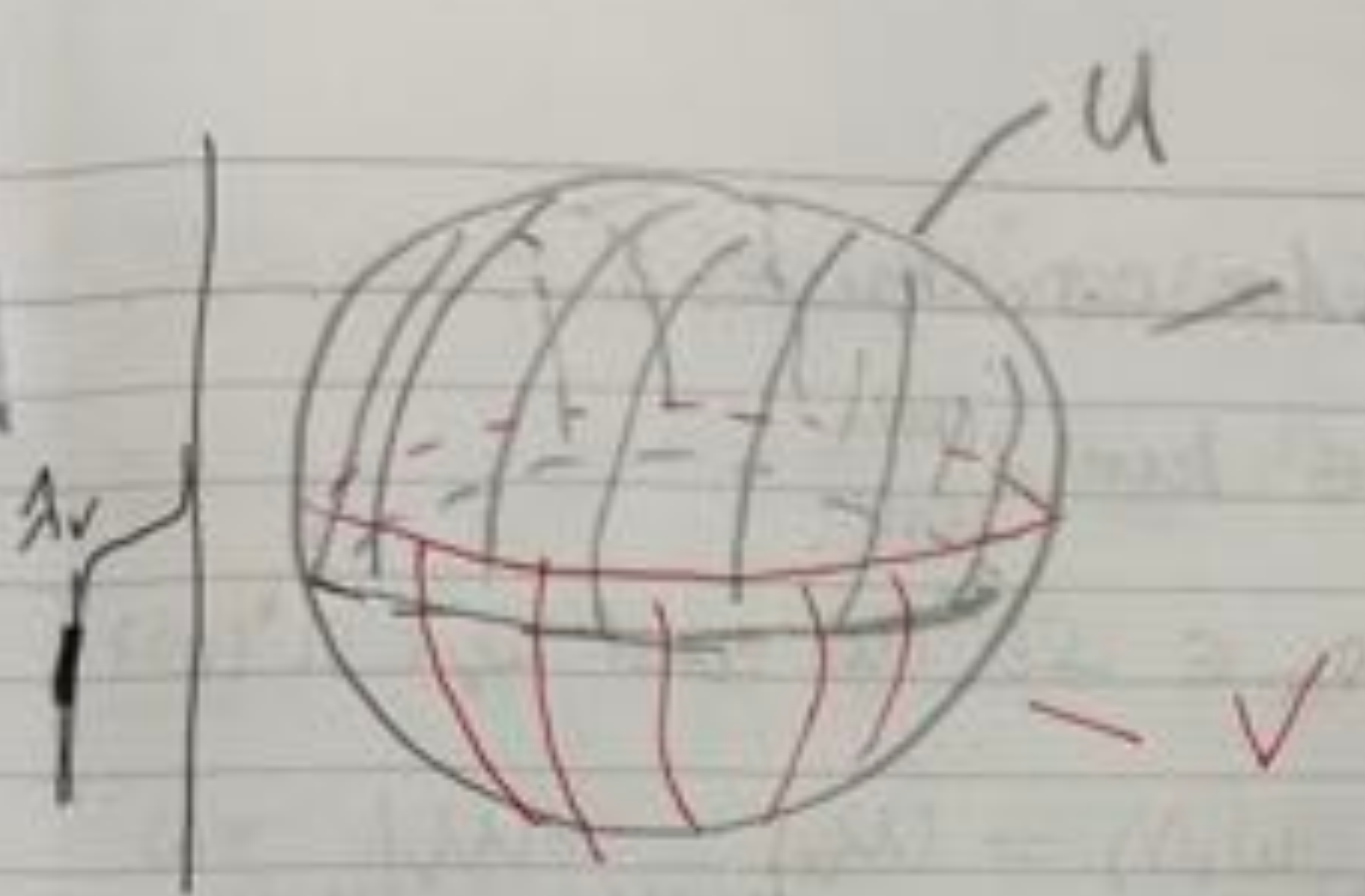
- β is surjective: Take $\rho \in \Omega^*(U \cap V)$ and a partition of unity $\lambda_U + \lambda_V = 1$ adapted to (U, V).

Then $\beta(\lambda_V \rho, -\lambda_U \rho) = (\lambda_V \rho)|_{U \cap V} - (\lambda_U \rho)|_{V \cap U}$

$$= (\lambda_V|_{U \cap V} + \lambda_U|_{U \cap V}) \rho = \rho.$$

where $\lambda_V \rho \in \Omega^*(U)$ and $\lambda_U \rho \in \Omega^*(V)$

More precisely: $\lambda_V \rho$ is ρ extended by zero on U:



This proves (1).

(2) Is a general fact from homological algebra if one has (1).

$$\begin{tikzcd}
 & 0 \arrow[r] & \ker d^{(k)}_{U\cup V} \arrow[r] \arrow[d] & \ker d^{(k)}_U \oplus d^{(k)}_V \arrow[r] \arrow[d] & \ker d^{(k)}_{U\cap V} \arrow[d] \arrow[ddll, bend left, "\delta"] & \\
 & 0 \arrow[r] & \Omega^k(U\cup V) \arrow[r, "\alpha^{(k)}"] \arrow[d, "d"'] & \Omega^k(U)\oplus\Omega^k(V) \arrow[r, "\beta^{(k)}"] \arrow[d, "d\oplus d"] & \Omega^k(U\cap V) \arrow[r] \arrow[d, "d"] & 0 \\
 & 0 \arrow[r] & \Omega^{k+1}(U\cup V) \arrow[r, "\alpha^{(k+1)}"] \arrow[d] & \Omega^{k+1}(U)\oplus\Omega^{k+1}(V) \arrow[r, "\beta^{(k+1)}"] \arrow[d] & \Omega^{k+1}(U\cap V) \arrow[r] \arrow[d] & 0 \\
 & & \operatorname{coker} d^{(k)}_{U\cup V} \arrow[r] & \operatorname{coker} d^{(k)}_U \oplus d^{(k)}_V \arrow[r] & \operatorname{coker} d^{(k)}_{U\cap V} \arrow[d] & \\
 & & & & 0 &
\end{tikzcd}$$

The sequence $\mathcal{Z}$ is exact.
(Snake lemma)

We have to construct δ:

Take $\omega \in \ker d^{(k)}_{U\cap V}$

$\overset{\beta \text{ surjective}}{\Longrightarrow}$ $\exists\, \omega_U \in \Omega^k(U)$ and $\omega_V \in \Omega^k(V)$:

$$\beta^{(k)}((\omega_U, \omega_V)) = \omega_U|_{U\cap V} - \omega_V|_{U\cap V} = \omega$$

Then $(d^{(k)}\omega_U, d^{(k)}\omega_V) \in \ker(\beta^{(k+1)})$ by commutativity of the right middle square.

$\Longrightarrow \exists\, \rho \in \Omega^{k+1}(U\cup V)$: $\alpha(\rho)$

$$\alpha(\rho) = (\rho|_U, \rho|_V) = (d^{(k)}\omega_U, d^{(k)}\omega_V)$$

(Note that $d^{(k+1)}(\rho) = 0$.)

We put $\delta\omega := \rho$.

And for $H^k(U\cap V) \xrightarrow{\delta} H^{k+1}(U\cup V)$

$$\delta([\omega]) := [\rho].$$

Exercise: δ is well-defined and the Mayer-Vietoris sequence is exact.

Example 3.70: (a) (spheres) S^n

$n=1$: $H^0_{dR}(S^1) \cong \mathbb{R}$ $H^1_{dR}(S^1) \cong \mathbb{R}$

$n=2$: $S^2 = U \cup V$

U = connected open sets

$= \{(x,y,z) \in S^2 \mid z > -\frac{1}{2}\}$

$V = \{(x,y,z) \in S^2 \mid z < \frac{1}{2}\}$

Mayer-Vietoris sequence

$$\begin{tikzcd}
0 \arrow[r] & \underset{\mathbb{R}}{H^0_{dR}(S^2)} \arrow[r, "\alpha^{(0)}"] & \underset{\mathbb{R}\,\oplus\,\mathbb{R}}{H^0_{dR}(pt) \oplus H^0_{dR}(pt)} \arrow[r, "\beta^{(0)}"] & \underset{\mathbb{R}}{H^0_{dR}(S^1)} \arrow[dlll, "\delta^{(0)}"] \\
H^1_{dR}(S^2) \arrow[r, "\alpha^{(1)}"] & \underset{0\,\oplus\,0}{H^1_{dR}(pt) \oplus H^1_{dR}(pt)} \arrow[r, "\beta^{(1)}"] & \underset{\mathbb{R}}{H^1_{dR}(S^1)} \arrow[dll, "\delta^{(1)}"] & \\
H^2_{dR}(S^2) \arrow[r] & 0 & &
\end{tikzcd}$$

We have $\ker \beta^{(0)} = \operatorname{im} \alpha^{(0)}$ is 1-dim.

$\Rightarrow$ $\operatorname{im} \beta^{(0)}$ is 1-dim. $\Rightarrow$ $\beta^{(0)}$ is surj.

$\Rightarrow \delta^{(0)} = 0 \Rightarrow \cancel{\delta} \, \alpha^{(1)}$ is injective

$\Rightarrow H^1_{dR}(S^2) = 0.$

Analogously $\delta^{(1)}$ is bijective

$\Rightarrow \quad H^2_{dR}(S^2) \cong \mathbb{R}.$

So $\quad H^i_{dR}(S^2) = \begin{cases} \mathbb{R}, \text{ if } i \in \{0,2\} \\ 0, \text{ if } i = 1 \text{ or } i > 2. \end{cases}$

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Analogously by induction for $n \geq 1$

$$H^i_{dR}(S^n) = \begin{cases} \mathbb{R}, \text{ if } i \in \{0,n\} \\ 0, \text{ if } i \in \mathbb{N} \setminus \{n \end{cases}$$

(b) Compute $\quad H^*_{dR}(\underbrace{S^1 \times \ldots \times S^1}_{n})$