(b) Every $\mathbb{R}$-algebra has a grading.

$$A = \bigoplus_{i=0}^{\infty} A_i \quad \text{with } A_i = \begin{cases} A, & i=0 \\ 0, & i>0 \end{cases}$$

(c) $\Omega^*(M)$ is a graded $\mathbb{R}$-algebra

$$\Omega^*(M) = \bigoplus_{i=0}^{\infty} \Omega^i(M)$$

because $\wedge(\Omega^i(M) \times \Omega^j(M)) \subseteq \Omega^{i+j}(M)$.

<u>Rk 3.45:</u> One also looks at $\mathbb{R}$-algebras with grading over $\mathbb{Z}$

$$A = \bigoplus_{i \in \mathbb{Z}} A_i.$$

<u>Ex:</u> $\mathbb{R}\langle X, X^{-1}\rangle = \bigoplus_{i=-\infty}^{\infty} \mathbb{R}X^i$

Def 3.46: (a) Let A be an $\mathbb{R}$-algebra. An $\mathbb{R}$-linear map $D: A \to A$ is called a derivation of A if it satisfies the Leibniz rule
$\forall a, b \in A: \quad D(ab) = aD(b) + D(a)b.$

(b) Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a graded $\mathbb{R}$-algebra. An $\mathbb{R}$-linear map $D: A \to A$ is called an antiderivation of A if it satisfies the Leibniz rule for antiderivation.
$\forall a \in A_i \ \forall b \in A_j$
$$D(ab) = D(a)b + (-1)^i a D(b).$$
(c) In (b), if $D \neq 0$ and $D(A_i) \subseteq A_{i+k} \forall i$ then $\deg(D) := k$.

Example 3.47: $D: \Omega^*(\mathbb{R}^n) \longrightarrow \Omega^*(\mathbb{R}^n)$

defined via

$$D\Big(\sum_{I \subseteq \mathbb{N}^{\leq n}} a_I \, dx^I\Big) := \sum_I da_I \wedge dx^I$$

is an antiderivation.

Proof: Linearity: Yes, because the wedge product is bilinear and the differential $d: C^\infty(\mathbb{R}^n) \longrightarrow \Omega^1(\mathbb{R}^n)$

$$\begin{array}{c} C^\infty(\mathbb{R}^n) \\ \parallel \\ \Omega^0(\mathbb{R}^n) \end{array}$$

is $\mathbb{R}$-linear.

Leibniz rule (for antiderivations):

We only need to consider elements of the form $a\,dx^I$, $a \in C^\infty(\mathbb{R}^n)$, $I \subseteq \mathbb{N}^{\leq n}$, because of the additivity of D.

$$D(a\,dx^I \wedge b\,dx^J) = D(ab\,\mathrm{sgn}(\sigma)\,dx^{I\cup J})$$

R

$$= \mathrm{sgn}(\sigma)\,d(ab) \wedge dx^{I \cup J}$$

$$= d(ab) \wedge dx^I \wedge dx^J$$

$$= ((da)b + a\,db) \wedge dx^I \wedge dx^J$$

Leibniz rule for the differential

$$= da \wedge dx^I \wedge b\,dx^J + a\,db \wedge dx^I \wedge dx^J$$

multi-linearity of wedge product.

$$= D(a\,dx^I) \wedge b\,dx^J + (-1)^{|I|} a\,dx^I \wedge db \wedge dx^J$$

$$= D(a\,dx^I) \wedge b\,dx^J + (-1)^{|I|} a\,dx^I \wedge D(b\,dx^J).$$

$\square$

D has more properties:

(i) $D\big|_{\Omega^0(\mathbb{R}^n)} = d$

(ii) $D \circ D = 0$

Pf: (i) ✓ by definition

(ii) $$D(D(a\,dx^I)) = D(da \wedge dx^I)$$

$$= D\Big(\sum_{i=1}^{n} \frac{\partial a}{\partial x_i} dx_i \wedge dx^I\Big)$$

$$= \sum_{i=1}^{n} d\Big(\frac{\partial a}{\partial x_i}\Big) \wedge dx_i \wedge dx^I$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 a}{\partial x_j \partial x_i} dx_j \wedge dx_i \wedge dx^I$$

$$\underset{\uparrow}{=} 0. \qquad \square$$

$dx_j \wedge dx_i = -dx_i \wedge dx_j$, $\dfrac{\partial^2 a}{\partial x_j \partial x_i} = \dfrac{\partial^2 a}{\partial x_i \partial x_j}$

<u>Def 3.48:</u> (Exterior derivative)

An <u>exterior derivative</u> on a mf M is an $\mathbb{R}$-linear map $D: \Omega^q(M) \to \Omega(M)$ which satisfies

(ED1) D is an antiderivation of degree 1

(ED2) $D \circ D = 0$

(ED3) $D|_{\Omega^0(M)} = d$ the differential.

<u>Theorem 3.49:</u> Let M be a mf. Then there exists a unique exterior derivative on M.

<u>Proof:</u> <u>Step 1:</u> We study the case $M = U \subseteq \mathbb{R}^m$ open

<u>Existence:</u> Example 3.47.

<u>Uniqueness:</u> Let D be the exterior derivative given by Example 3.47. and let D'

be a second one. Induction on the degree:

(BC): $\omega \in \Omega^0(U) = C^\infty(U)$: $D(\omega) = d\omega = D'(\omega)$ by (ED3).

(IS): Let $D(\omega) = D'(\omega)$ $\forall \omega \in \bigoplus_{i=0}^{k} \Omega^i(U)$.

Take $a\, dx^I$ with $a \in C^\infty(U)$ and $I \subseteq \mathbb{N}^{\leq m}$ s.t. $|I| = k+1$.

Then
$$D'(a\, dx^I) \underset{(ED4)}{=} D'(a) \wedge dx^I + (-1)^0 a \wedge D'(dx^I)$$
$$= da \wedge dx^I + a D'(dx^I)$$
$$\underset{(ED3)}{=} D(a\, dx^I) + a D'(dx^I).$$

<u>To show:</u> $D'(dx^I) = 0$.

<u>Pf:</u>
$$D'(dx^I) \underset{I = \{i_1 < i_2 < \ldots < i_{k+1}\}}{=} D'(D(x_{i_1} dx^{I \setminus \{i_1\}}))$$
$$\underset{(IH)}{=} D'(D'(x_{i_1} dx^{I \setminus \{i_1\}})) \underset{(ED2)}{=} 0 \;\square$$

Step 2: (local property of exterior derivations)

Let D be an exterior derivation on M and $\omega \in \Omega^k(M)$ and $P \in M$ s.t. $\omega \equiv 0$ around P, i.e. $\exists\, U \subseteq M$ open s.t. $P \in U$ and $\omega|_U = 0$.

Then $D(\omega)|_U = 0$ and $D(\omega)(P) = 0$.

Pf: W.l.o.g. U is part of a chart (φ, U). Take $\lambda : M \to [0,1]$ C^∞ s.t. $\lambda \equiv 0$ around P and $\lambda \equiv 1$ on $M \setminus U$, s.t. $\lambda\omega = \omega$.

R

Then $D(\omega) = D(\lambda\omega) = d\lambda \wedge \omega + (-1)^0 \lambda \wedge D(\omega)$.

$\Rightarrow D(\omega)(Q) = (d\lambda)(Q) \wedge \underbrace{\omega(Q)}_{=0} + \underbrace{\lambda(Q)}_{0} D(\omega)(Q)$

for Q near around P. $\square$

Step 3: Let $U \subseteq M$ open and D be an exterior derivative on M. We define $D_U : \Omega^*(U) \to \Omega^*(U)$ as follows:

$\omega \in \Omega^*(U)$

$$D_U(\omega)(p) := D(\lambda\omega)(p)$$

where λ ~~is a~~ $\in C^\infty(M, [0,1])$ satisfies $\lambda \equiv 1$ around p and $\lambda \equiv 0$ on $M \setminus V$ s.t.

$$p \in V \subseteq \underbrace{\bar{V}}_{\text{compact}} \subset U$$

("bump function near around p")

Then D_U is well-defined by Step 2.

Exercise: D_U is an exterior derivative on U.

Step 4: (Uniqueness)

Let D and D' be exterior derivatives on M. Take $(\varphi, U) \in \mathcal{A}_\infty$ and $p \in U$. Then for $\omega \in \Omega^*(M)$ we

$$\text{have} \quad D(\omega)(p) \overset{\substack{\text{Def. of } D_U \\ \downarrow}}{=} D_U(\omega|_U)(p)$$

$$\underset{\substack{\uparrow \\ D_U = D'_U \text{ by} \\ \text{Step 1 (uniqueness)}}}{=} D'_U(\omega|_U)(p) \underset{\substack{\uparrow \\ \text{Def of } D'_U}}{=} D'(\omega)(p).$$

<u>Step 5</u>: (Existence)

For every chart (φ, U) we have a unique exterior derivative $D_U : \Omega^0(U) \to \Omega^1(U)$.

We define $D : \Omega^0(M) \to \Omega^1(M)$ via

$$D(\omega)(p) := D_U(\omega|_U)(p).$$

for $\omega \in \Omega^0(M)$ and $p \in M$ and $(\varphi, U) \in \mathcal{A}_\infty$ with $p \in U$.

Exercise: D is an exterior derivative on M. $\square$

Def 3.50: Let M be a manifold.
The complex

$$0 \xrightarrow{d^{(-1)}} \Omega^0(M) \xrightarrow{d^{(0)}} \Omega^1(M) \xrightarrow{d^{(1)}} \Omega^2(M) \to \cdots \xrightarrow{d^{(m-1)}} \Omega^m(M) \to 0 = \Omega^{m+1}(M)$$

is called the "de Rham complex of M".
If there is no reason for confusion we write d instead of $d^{(i)}$.
Here d^i is the restriction of the exterior derivative of M to $\Omega^i(M)$.

A smooth k-form $\omega \in \Omega^k(M)$ is called (i) <u>closed</u> if $d^{(k)}\omega = 0$

(ii) <u>exact</u> if $\exists\, \tau \in \Omega^{k-1}(M)$: $d^{(k-1)}\tau = \omega$.

We call the $\mathbb{R}$-v.s.

$$H^k_{dR}(M) := \ker(d^{(k)}) / \operatorname{im}(d^{(k-1)})$$

the <u>k-th de Rham cohomology of M</u>, $k = 0, 1, 2, \dots, m$.

End of Lecture 4.4.2023

<u>Example 3.51:</u> Let M be a mf with connected components M_i, $i \in I$. (We know that $|I| \le |M|$ by our convention 1.10)

Then $H^0_{dR}(M) \cong \mathbb{R}^{|I|}$, because a function $f \in C^\infty(M)$ with $df = 0$ is constant on every connected component.

<u>Example 3.52:</u> (de Rham cohomology of S^1)

$$
\begin{array}{ccccc}
0 \to \Omega^0(S^1) & \xrightarrow{d^{(0)}} & \Omega^1(S^1) & \longrightarrow & 0 \\
\| & & \| & & \\
C^\infty(S^1) & & C^\infty(S^1)\,(-y\,dx + x\,dy)|_{S^1} & &
\end{array}
$$

$H^0_{dR}(S^1) \cong \mathbb{R}$, because S^1 is connected

$H^1_{dR}(S^1) = ?$

Consider $\Phi: \Omega^1(S^1) = \ker(d^{(1)}) \to \mathbb{R}$

$$\Phi(\omega) := \int_0^{2\pi} \omega_{c(t)}(c'(t))\,dt$$

(Reminder: $c'(t) = (T_t c)\left(\frac{\partial}{\partial t}\right) = c_1'(t)\frac{\partial}{\partial x}(c(t))$

$+ c_2'(t)\frac{\partial}{\partial y}(c(t))$

by Remark 3.21.)

We have $\operatorname{im}(d^{(0)}) \subseteq \ker \Phi$, because

$$\int_0^{2\pi} (df)_{c(t)}(c'(t))\,dt$$

$$= \int_0^{2\pi} (f \circ c)'(t)\,dt = f(c(2\pi)) - f(c(0)) = 0,$$

and Φ is surjective, because for $\omega = (-y\,dx + x\,dy)|_{S^1}$ we get

$$\Phi(\omega) = \int_0^{2\pi} -c_2(t)\,dx(c'(t)) + c_1(t)\,dy(c'(t))\,dt$$

$$= \int_0^{2\pi} -c_2(t)c_1'(t) + c_1(t)c_2'(t)\,dt$$

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t\,dt = 2\pi.$$

Let $\omega \in \ker \Phi$ and put

$$f(c(t)) := \int_0^t \omega_{c(s)}(c'(s))\,ds.$$

Then

$$(df)_{c(t)}(c'(t)) = (df)_{c(t)}(T_t c)\left(\frac{\partial}{\partial t}\right)$$

$$\underset{\substack{\uparrow \\ \text{chain rule}}}{=} d(f\circ c)_t\left(\frac{\partial}{\partial t}\right)$$

$$\underset{\substack{\uparrow \\ \text{notation}}}{=} (f\circ c)'(t) = \left(\int_0^t \omega_{c(s)}(c'(s))\,ds\right)'$$

$$= \omega_{c(t)}(c'(t)).$$

$c'(t) = c_1'(t)\frac{\partial}{\partial x}(c(t)) + c_2'(t)\frac{\partial}{\partial y}(c(t))$ generates $T_{c(t)}S^1$. Thus $(df)_{c(t)} = \omega_{c(t)}$.

c is surjective. Thus $df = \omega$.

So

$$\begin{tikzcd} \ker d^{(1)} \arrow[r, "\Phi", "\mathbb{Q}"'] \arrow[d] & \mathbb{R} \\ H^1_{dR}(S^1) = \ker d^{(1)}/\ker\Phi \arrow[ur, "\exists!\,\bar{\Phi}", "\sim"'] & \end{tikzcd}$$

End Sec 3.5!

Example 3.53: (Poincaré lemma)

$n \in \mathbb{N}_0$. Then we have

$$H^k_{dR}(\mathbb{R}^n) \simeq \begin{cases} \mathbb{R}, & k=0 \\ 0, & k>0. \end{cases}$$

Proof: $H^0_{dR}(\mathbb{R}^n) \simeq \mathbb{R}$, because $\mathbb{R}$ is connected

$k \geq 1$: Take $\omega = a\, dx_{i_1} \wedge \ldots \wedge dx_{i_k} \in \Omega^k(\mathbb{R}^n)$

such that $d\omega = 0$, i.e.

$$\sum_{j \notin \{i_1, \ldots, i_k\}} \frac{\partial a}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} = 0$$

Thus $\frac{\partial a}{\partial x_j} \equiv 0 \quad \forall j \notin \{i_1, \ldots, i_k\}$,

i.e. $a = a(\cancel{x_{I \setminus \{i_1, \ldots, i_k\}}}) = a(x_{i_1}, \ldots, x_{i_k})$

Define $b(x_{i_1}, \ldots, x_{i_k}) := \int_0^{x_{i_1}} a(s, x_{i_2}, \ldots, x_{i_k})\, ds$

Then $d(b\, dx_{i_2} \wedge \dots \wedge dx_{i_k}) = \omega$. $\square$

We have four tools to compute de Rham cohomology groups.

1) Poincaré lemma
2) integration, to show that certain cocycles are ~~not~~ not coboundaries
3) Mayer-Vietoris sequence
4) Homotopy.

<u>Notation 3.54:</u> Let M be a mf.

$$0 \xrightarrow{d^{(-1)}} \Omega^0(M) \xrightarrow{d^{(0)}} \Omega^1(M) \xrightarrow{d^{(1)}} \dots \xrightarrow{d^{(m-2)}} \Omega^{m-1}(M) \xrightarrow{d^{(m-1)}} \Omega^m(M)$$

For $k = 0, \dots, m$

$Z^k_{dR}(M) := \ker d^{(k)}$
= set of closed k-forms
= set of <u>cocycles of degree k</u>

$B^k_{dR}(M) = \operatorname{im} d^{(k-1)}$
= set of exact k-forms
= set of <u>coboundaries of degree k</u>.

$$H^k_{dR}(M) = \frac{Z^k_{dR}(M)}{B^k_{dR}(M)} = k\text{-th de Rham cohomology}$$

The exterior derivative behaves well under smooth maps.

Def 3.55: Let $f \in C^\infty(M,N)$ and $\omega \in \Omega^k(N)$.
We define $f^*\omega \in \Omega^k(M)$ via

$$(f^*\omega)_p(v_1,\dots,v_k) :=$$

$$\omega_{f(p)}(T_pf(v_1),\dots,T_pf(v_k))$$

It is called the pullback of ω under f.

$$f^*: \Omega^*(N) \longrightarrow \Omega^*(M)$$
is an $\mathbb{R}$-algebra homomorphism
(Exercise!)

Lemma 3.56: Let $f \in C^\infty(M,N)$.

Then the exterior derivatives d_M and d_N satisfy
$$d_M \circ f^* = f^* \circ d_N$$

$$\begin{array}{ccccccc} 0 \to \Omega^0(M) & \xrightarrow{d_M} & \Omega^1(M) & \xrightarrow{d_M} & \Omega^2(M) & \xrightarrow{d_M} \\ \uparrow f^* & G & \uparrow f^* & G & \uparrow f^* & \\ 0 \to \Omega^0(N) & \xrightarrow{d_N} & \Omega^1(N) & \xrightarrow{d_N} & \Omega^2(N) & \xrightarrow{d_N} \end{array}$$

<u>Proof:</u> By the local property of f^*

($\omega_1 = \omega_2$ around $f(p) \Rightarrow f^*\omega_1 = f^*\omega_2$ around

we can reduce to forms cptly supported in a chart.

Since f^* is an $\mathbb{R}$-algebra homomorphism we can reduce to forms in ~~[illegible]~~ $\Omega^0(N) \cup B^1(N)$

$g \in \Omega^0(N)$: $p \in M$, $v \in T_pM$

$$f^*(d_N g)_p(v) = (d_N g \circ T_p f)(v)$$

$$\underset{\text{Chain rule}}{=} d(g \circ f)_p(v) = d_M(f^*(g))_p(v)$$

$dg \in B^1(N)$: $\quad p \in M,\ v \in T_pM$

$$f^*(d_N(dg)) = f^*0 = 0 \in \Omega^2(M)$$

$$d_M(f^*(dg)) \underset{\substack{\uparrow \\ \text{Part 1}}}{=} d_M(d_M(f^*(g))) = 0$$

□

<u>Corollary 3.57:</u> Let $f \in C^\infty(M,N)$.

Then f^* induces a map

$H^k_{dR}(N) \longrightarrow H^k_{dR}(M)$ for all

$k = 0, 1, 2, \dots$.

<u>Proof:</u> By Lemma 3.56 we have

for $k \geq 0$: $f^*(\operatorname{im} d_N^{(i)}) \subseteq \operatorname{im} d_M^{(i)}$

and $f^*(\ker d_N^{(i)}) \subseteq \ker d_M^{(i)}$.

For the second, see:

$$\omega \in \ker d_N^{(i)} \Rightarrow (d_M \circ f^*)(\omega) = f^*(d_N(\omega)) = f^*(0) = 0. \; \square$$

Example 3.58:

$$S^n \overset{\iota}{\subseteq} \mathbb{R}^n \setminus \{\underline{0}\} \xrightarrow{f} S^n$$

$$f(\underline{x}) := \frac{1}{|\underline{x}|_2} \underline{x}.$$

$\Rightarrow \quad H^k(S^n) \xrightarrow{f^*} H^k(\mathbb{R}^n \setminus \{\underline{0}\}) \xrightarrow{\iota^*} H^k(S^n)$

and $\quad \iota^* \circ f^* = (f \circ \iota)^* = (id_{S^n})^*$
$\qquad\qquad = id_{H^k(S^n)}$

on cohomology.

$\Rightarrow \quad \dim_{\mathbb{R}} H^k(S^n) \leq \dim_{\mathbb{R}} H^k(\mathbb{R}^n \setminus \{\underline{0}\})$

$\Rightarrow \quad H^k(S^n) \simeq H^k(\mathbb{R}^n \setminus \{\underline{0}\}) \quad \forall k \in \mathbb{N}_0.$
we see "$\geq$" later

We now come to the homotopy invariance.

Lemma 3.59: Let M be a mf.
Consider the two maps
$$f_i : M \to M \times [0,1], \quad i = 1, 0$$
$$f_i(p) := (p, i).$$
~~Let ...~~

Then $\exists$ linear map
$L : \Omega^*(M \times [0,1]) \longrightarrow \Omega^*(M)$ of degree -1 such that

$$f_1^* - f_0^* = d_M \circ L + L \circ d_{M\times[0,1]}$$

<u>Remark 3.60:</u> 1) picture

$$\begin{tikzcd}
\Omega^{k-1}(M\times[0,1]) \arrow[r] & \Omega^{k}(M\times[0,1]) \arrow[r] \arrow[dl, "L^{(k)}"'] & \Omega^{k+1}(M\times[0,1]) \arrow[dl, "L^{(k+1)}"'] \\
\Omega^{k-1}(M) \arrow[r] & \Omega^{k}(M) \arrow[r] & \Omega^{k}(M)
\end{tikzcd}$$

$$d_M^{(k-1)} \circ L^{(k)} + L^{(k+1)} \circ d_{M\times[0,1]}^{(k)} = f_1^* - f_0^*$$

on $\Omega^k(M\times[0,1])$.

2) Take $\omega \in Z^k(M\times[0,1])$.

Then $(f_1^* - f_0^*)\,\omega = (d_M^{(k-1)} L)(\omega)$

$\in B^{k-1}(M)$.

i.e. $f_1^*\omega$ and $f_0^*\omega$

differ by an exact form.

$\Rightarrow$ On $H^*(M\times[0,1])$
$$\| $$
$$\bigoplus_k H^k(M\times[0,1])$$

f_1^* and f_0^* agree.

$$f_i^*: H^*(M\times[0,1]) \to H^*(M)$$

$i=0,1$.

End of Lecture 7.4.2023

Proof of Lemma 3.59:

Step 1:
$$M \xrightarrow{\iota} M\times[0,1] \xrightarrow{\pi} M$$
$$p \mapsto (p,1) \quad (p,s) \mapsto p.$$
$$\pi\circ\iota = id_M$$

$$\Rightarrow \pi^*: \Omega^*(M) \hookrightarrow \Omega^*(M\times[0,1])$$

Step 2: For $\omega\in\Omega^k(M\times[0,1])$, $k>0$ we have the decomposition
$$\omega = a_1\omega_1 + a_2\omega_2\wedge dt$$