Under $\mathbb{R}^2 \otimes \mathbb{R}^2 \xrightarrow[\sim]{\Phi} M_2(\mathbb{R})$
$v \otimes w \longmapsto v w^t$

the elementary tensors correspond to matrixes of rk ≤ 1, but

$$\Phi(e_1 \otimes e_2 + e_2 \otimes e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

has rank 2.

A tensor is a finite sum (!) of elementary tensors.

(b) Decompose $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ using the smallest number of elementary tensors.

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \Phi\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) + \Phi\left(\begin{pmatrix} 0 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$= \Phi\left((e_1 + e_2) \otimes (e_1 + e_2) + 2 e_2 \otimes e_2\right)$$

Def 3.19: Let V be finite dimensional real vector space and $k \in \mathbb{N}$. Let M be the real subspace of $\underbrace{V \otimes \ldots \otimes V}_{k} = V^{\otimes k}$

generated by the ~~elements~~ set

$\{v_1 \otimes v_2 \otimes \ldots \otimes v_k \mid v_i \in V,$

$\exists\, i,j \;\; i \neq j : v_i = v_j \}$

$\Lambda^k V := V^{\otimes k} / M$ is called

the k-th exterior ~~product~~ power of V.

We write $v_1 \wedge \ldots \wedge v_k$ for the class $[v_1 \otimes \ldots \otimes v_k]_M$.

Prop 3.20: Let V be a finite dim real vector space. Then $\Lambda^k(V^*)$ is $\mathbb{R}$-linear isomorphic to $Alt^k(V)$ via a map

$$\Phi : Alt^k(V) \longrightarrow \Lambda^k(V^*)$$

which satisfies

$$\underbrace{\Phi(f_1 \wedge \dots \wedge f_k)}_{\substack{\text{wedge product} \\ \text{for alternating} \\ \text{forms}}} = \underbrace{f_1 \wedge \dots \wedge f_k}_{\substack{\| \\ [f_1 \otimes \dots \otimes f_k]}}$$

Proof: Define Φ as above on a basis of $Alt^k(V)$. Show that Φ is injective. (Exercise!) $\square$

Remark 3.21: (transformation matrix)

Let M and N be manifolds and $f: M \to N$ C^∞. Take charts (φ, U) $(x_1, \dots, x_m)$ and (ψ, V) $(y_1, \dots, y_n)$ of M and N resp. s.t. $f(U) \subseteq V$. Then for $p \in U$:

$$\left((T_p f)\left(\frac{\partial}{\partial x_1}(p)\right), \dots, (T_p f)\left(\frac{\partial}{\partial x_m}(p)\right) \right)$$

$$= \left(\frac{\partial}{\partial y_1}(p), \dots, \frac{\partial}{\partial y_n}(p) \right) A(p)$$

for $A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}, & \dots, & \frac{\partial f_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}, & \dots, & \frac{\partial f_n}{\partial x_m} \end{pmatrix}$

where $f_i := \psi_i \circ f : U \longrightarrow \mathbb{R}$.

Proof: $(T_p f)\left(\frac{\partial}{\partial x_i}(p)\right) = (T_p f)([p, \varphi, e_i])$

$= [f(p), \psi, \underbrace{D(\psi \circ f \circ \varphi^{-1})(\varphi(p)) e_i}_{\text{Jacobi matrix of } \psi\circ f\circ \varphi^{-1} \text{ at } \varphi(p)}]$

$= \left[f(p), \psi, \sum_{j=1}^{n} e_j \underbrace{\frac{\partial(\psi \circ f \circ \varphi^{-1})_j}{\partial x_i}}_{\text{partial derivatives}} (\varphi(p))\right]$

$= \left[f(p), \psi, \sum_{j=1}^{n} \underbrace{\frac{\partial(\psi_j \circ f)}{\partial x_i}}_{\text{derivation w.r.t. } x_i} (p) e_j\right]$

$$= \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_i}(p) \, [f(p), \varphi, e_j]$$

$$= \sum_{j=1}^{n} \underbrace{\frac{\partial}{\partial y_j}\Big|_{f(p)}}_{\in T_{f(p)}N} \cdot \underbrace{\frac{\partial f_j}{\partial x_i}(p)}_{\in \mathbb{R}} . \qquad \square$$

Def 3.22: Let V be a f.d. $\mathbb{R}$-v.s.

(a) A k-tensor is a k-linear function $f : V \times \cdots \times V \longrightarrow \mathbb{R}$

(i.e. an element of $V^* \otimes \cdots \otimes V^*$ after identifying ~~Ltr~~ $\mathrm{Mult}^k(V, \mathbb{R})$ with $(V^*)^{\otimes k}$.)

(b) An alternating k-form $f \in \mathrm{Alt}^k(V)$ is also called alternating k-tensor

(We identify $\Lambda^k(V^*)$ with $\mathrm{Alt}^k(V)$)

It is also called k-covector.

<u>Notation 3.23:</u> Let M be a mf. and $p \in M$.

We write T_p^*M for $(T_pM)^*$, the dual vector space of T_pM.

<u>Def 3.24:</u> Let M be a mf.. A <u>k-covector field</u> on M (also <u>differential k-form</u> on M, k-form on M)

is a map $\omega : M \longrightarrow \Lambda^k(T^*M)$ such that $\omega(p) \in \Lambda^k(T_p^*M)$, for all $p \in M$.

<u>Example 3.25:</u> Let (φ, U) be a chart on M.

$$\varphi = (x_1, \dots, x_m)$$

Let $\omega_1(p), \dots, \omega_m(p)$ be the dual basis of $\frac{\partial}{\partial x_1}(p), \dots, \frac{\partial}{\partial x_m}(p)$.

Then $\omega_i : U \longrightarrow \Lambda^1(T^*U) = \Lambda^1(T^*M)$, where $\Lambda^1(T^*M) = T^*M$, are 1-covector fields on U.

We can obtain $\omega_1, \dots, \omega_m$ in a different way.

<u>Def 3.26</u>: Let $f \in C^\infty(M, \mathbb{R})$. Using the derivative $Tf : TM \longrightarrow T\mathbb{R}$ of f, we obtain a 1-form as follows

$$
\begin{tikzcd}
TM \arrow[r, "Tf"] \arrow[rrr, bend right, "df"'] & T\mathbb{R} \arrow[r, phantom, "\simeq"] & \mathbb{R} \times \mathbb{R} \arrow[r, "pr_2"] & \mathbb{R}
\end{tikzcd}
$$

$$[p, id, v] \mapsto (p, v)$$

The map $df : M \longrightarrow T^*M$ is called <u>differential of f</u>

<u>Remark 3.27:</u>

(i) We have in coordinates $\varphi = (x_1, \dots, x_n)$:

$$(T_p f)(X) = (df)_p(X) \cdot \frac{\partial}{\partial t}(f(p)),$$

$X \in T_p M$, using the chart $(id_{\mathbb{R}}, \mathbb{R})$ on $\mathbb{R}$

$$id_{\mathbb{R}} = (t)$$

(ii) Given coordinates $\varphi = (x_1, \dots, x_m)$ we have $x_1, \dots, x_m \in C^\infty(U, \mathbb{R})$.

Thus $dx_1, \dots, dx_m$ are 1-forms on U.

<u>Claim:</u> $(dx_1)_p, \dots, (dx_m)_p$ is the dual basis of ~~the~~ $\frac{\partial}{\partial x_1}(p), \dots, \frac{\partial}{\partial x_m}(p)$.

<u>Proof:</u>

$$\left(T_p x_i \left(\frac{\partial}{\partial x_1}(p)\right), T_p x_i \left(\frac{\partial}{\partial x_2}(p)\right), \dots, T_p x_i \left(\frac{\partial}{\partial x_m}(p)\right) \right)$$

$$= \frac{\partial}{\partial t}\Big|_{x_i(p)} \left(\frac{\partial x_i}{\partial x_1}(p), \dots, \frac{\partial x_i}{\partial x_m}(p) \right)$$

by Remark 3.21.

Now $\frac{\partial x_i}{\partial x_j}(p) = \delta_{ij}$, $p \in U$. □

Def 3.28: (The cotangent bundle on M)

Let M be a mf. $T^*M := \bigsqcup_p T^*_p M$

$$= \{(p,f) \mid p \in M \text{ and } f \in T^*_p M\}$$

is called the <u>cotangent bundle on M</u>

We define a differential structure on T^*M using charts $(\varphi = (x_1, \dots, x_m), U)$ of M via:

$$T^*M \supseteq T^*U \xrightarrow{\sim} U \times \mathbb{R}^m$$

$$\left(p, \sum_{i=1}^{m} \lambda_i (dx_i)_p\right) \longmapsto (\varphi(p), \lambda_1, \dots, \lambda_m)$$

Which matrix is the transformation matrix for a coordinate change?

(Exercise!)

We have more vector bundles:

- The tensor bundle of type (p,q):

$$(TM)^{\otimes p} \otimes (T^*M)^{\otimes q} \longrightarrow M$$

- The k-th exterior power of the cotangent bundle on M:

$$\Lambda^k(T^*M) \longrightarrow M.$$

The tangent bundle is the tensor bundle of type $(1,0)$.

End of Lecture 28.3.23

Def 3.29: We call a differential k-form ω smooth if $\omega \in C^\infty(M, \Lambda^k(T^*M))$.

(Recall: For $\Lambda^k(T^*M)$ a local frame is given by

$$\{ dx_{i_1} \wedge \dots \wedge dx_{i_k} \mid 1 \leq i_1 < \dots < i_k \leq m \}$$

ordered in a lexicographic way

(Ex: $dx_1 \wedge dx_3 \wedge dx_5 \wedge dx_6$

$< dx_1 \wedge dx_3 \wedge dx_4 \wedge dx_7$)

<u>Notation 3.30:</u> We write $\Omega^k(M)$ for the set of smooth differential k-forms, $k = 0, 1, 2, \ldots$

Here $\Omega^0(M) = C^\infty(M, \mathbb{R})$

<u>Example 3.31:</u> (a) Find $\omega \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$ which satisfies

$$(*) \begin{cases} \omega\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) \equiv 1 \text{ and} \\ \omega\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \equiv 0. \end{cases}$$

Solution: Ansatz

$$\omega = a\,dx + b\,dy$$

$$(*) \iff \begin{array}{l} -ay + bx \equiv 1 \\ \wedge \quad ax + by \equiv 0 \end{array}$$

$$\iff a = \frac{-y}{x^2+y^2} \wedge b = \frac{+x}{x^2+y^2}.$$

Thus the solution is

$$\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

(b) We have a map

$$d : \Omega^0(M) \longrightarrow \Omega^1(M)$$
$$f \longmapsto df.$$

Is there an $f \in \Omega^0(M)$ $(M = \mathbb{R}^2 - \{\underline{0}\})$
such that $df = \omega$?

If yes then:

$$\nabla f \cdot \begin{pmatrix} -y & x \\ x & y \end{pmatrix} = (1, 0)$$

$$\Longleftrightarrow \quad \nabla f = (-y, +x) \cdot \frac{1}{x^2+y^2}$$

$\uparrow$

$$\begin{pmatrix} -y & x \\ x & y \end{pmatrix}^{-1} = \begin{pmatrix} y & -x \\ -x & -y \end{pmatrix} \frac{1}{-x^2-y^2}$$

<u>Another way</u>: $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy.$

$\Rightarrow \frac{\partial f}{\partial x} = \frac{-y}{x^2+y^2}$ and $\frac{\partial f}{\partial y} = \frac{x}{x^2+y^2}$.

Put $g(t) := f(\cos t, \sin t)$. and $c(t) = (\cos t, \sin t)$,
$t \in \mathbb{R}$.

$$\Rightarrow \int_0^{2\pi} g'(t)\, dt = \int_0^{2\pi} \nabla f(c(t)) c'(t)\, dt$$
$$= \int_0^{2\pi} 1\, dt = 2\pi$$

and $\int_0^{2\pi} g'(t)\, dt = g(2\pi) - g(0) = 0.$ ↯

(c) We have another map

$$d : \Omega^1(\mathbb{R}^2 \setminus \{\underline{0}\}) \longrightarrow \Omega^2(\mathbb{R}^2 \setminus \{\underline{0}\})$$

given by

$$d(a\,dx + b\,y) := da \wedge dx + db \wedge y$$

So we have

$$\Omega^0(\mathbb{R}^2 \setminus \{\underline{0}\}) \xrightarrow{d_0} \Omega^1(\mathbb{R}^2 \setminus \{\underline{0}\}) \xrightarrow{d_1} \Omega^2(\mathbb{R}^2 \setminus \{\underline{0}\})$$

The "de Rham complex of $\mathbb{R}^2 \setminus \{\underline{0}\}$".

(d) We have $d^2 = 0$.

<u>Proof:</u>

$$d(df)$$
$$\shortparallel$$
$$d\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)$$
$$\shortparallel$$
$$d\left(\frac{\partial f}{\partial x}\right) \wedge dx + d\left(\frac{\partial f}{\partial y}\right) \wedge dy$$
$$\shortparallel$$
$$\left(\frac{\partial^2 f}{\partial x^2}dx + \frac{\partial^2 f}{\partial y \partial x}dy\right) \wedge dx + \left(\frac{\partial^2 f}{\partial x \partial y}dx + \frac{\partial^2 f}{\partial y^2}dy\right) \wedge dy$$
$$\shortparallel$$
$$\frac{\partial^2 f}{\partial y \partial x}dy \wedge dx + \frac{\partial^2 f}{\partial x \partial y}dx \wedge dy$$
$$\shortparallel$$
$$0.$$

$\square$

(c2) The complex is not exact in the middle, because

$$\omega = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

is in the kernel of d_1, but not in the image of d_0.

<u>Pf:</u> · By (b): $\omega \notin \operatorname{im} d_0$.

· $$d\omega = d\left(\frac{-y}{x^2+y^2}\right)\wedge dx + d\left(\frac{x}{x^2+y^2}\right)\wedge dy$$

$$= \left(\frac{+2xy}{(x^2+y^2)^2}\,dx + \frac{y^2-x^2}{(x^2+y^2)^2}\,dy\right)\wedge dx$$

$$+ \left(\frac{y^2-x^2}{(x^2+y^2)^2}\,dx + \frac{(-2yx)}{(x^2+y^2)^2}\,dy\right)\wedge dy$$

$$= \frac{y^2-x^2}{(y^2+y^2)^2}\left(dy\wedge dx + dx\wedge dy\right)$$

$$= 0 \quad \square$$

Thus M has nontrivial first de Rham cohomology

$$H^1_{dR}(\mathbb{R}^2\setminus\{0\}) := \ker d_1 / \operatorname{im} d_0 \neq \{[0]\}$$

In fact $H^1_{dR}(\mathbb{R}^2 \setminus \{0\}) \overset{\Phi}{\simeq} \mathbb{R}$ via

$$[\omega] \longmapsto \frac{1}{2\pi}\int_0^{2\pi} \omega(c'(t))\,dt.$$

Proof: Surjectivity: $1 \in \operatorname{im}\Phi$ by!!

Injectivity. let $\omega \in \ker\Phi$.

Put $f(p) := \int_{c_p} \omega(c_p'(t))\,dt$

where

c_p is a curve from $(1,0)$ to p going first
$\ \| \\ re^{i\varphi}$

to $e^{i\varphi}$ via c and then to p on a segment.

Instead of c we could have taken c^{-1} to define f_1 ~~so~~ because $\omega \in \ker \Phi$:

$$\left(\int_{c_p} \omega(c_p'(t))\,dt - \int_{\tilde{c}} \omega(\tilde{c}'(t))\,dt \right.$$

$$\left. = \int_c \omega(c'(t))\,dt = 0. \right)$$

Now it is an easy exercise to show that $df = \omega$ $\square$

(d) Find $\Omega^1(S^1)$.

We consider S^1 as a subm of $\mathbb{R}^2$

$$\underset{(x_p, y_p)}{T_p} S^1 \subseteq T_p \mathbb{R}^2$$

$$= \left\{ a \frac{\partial}{\partial x}(p) + b \frac{\partial}{\partial y}(p) \;\middle|\; \begin{array}{l} a x_p + b y_p = 0 \\ a, b \in \mathbb{R} \end{array} \right\}$$

$$= \mathbb{R} \left(-y_p \frac{\partial}{\partial x}(p) + x_p \frac{\partial}{\partial y}(p) \right).$$

$$\Omega^1(S^1) = C^\infty(S^1) \left((-y\,dx + x\,dy)|_{TS^1} \right).$$

Remark 3.40: Let M be the Möbius strip and N be the submanifold obtained by removing a cutting segment from M.

$N = M \setminus$ segment

Then $\Omega^1(M) \xrightarrow{res} \Omega^1(N)$ is not surjective.

We have to introduce the exterior derivative and the pullback of differential form under a C^∞-map rigorously.

Def 3.41: A quadruple $(A, +: A\times A \to A, \cdot: \mathbb{R}\times A \to A, \odot: A\times A \to A)$ is called an $\mathbb{R}$-algebra if

(A1) $(A, +, \cdot)$ is an $\mathbb{R}$-v.s.

(A2) $(A, +, \odot)$ is a ring (possibly not commutative) and

(A3) $\forall_{r_1, r_2 \in \mathbb{R}} \forall_{a_1, a_2 \in A}$ $(r_1 a_1) \odot (r_2 a_2) = (r_1 r_2)\cdot(a_1 \odot a_2)$.

Ex. 3.42: (a) $\mathbb{R}$ is an $\mathbb{R}$-algebra
$(r_1 \odot r := r_1 r)$

(b) $M_n(\mathbb{R})$ is an $\mathbb{R}$-algebra using · $r \cdot \underbrace{A}_{(a_{ij})} := (r a_{ij})_{ij}$.

· $A \odot B = AB$.

(c) $C^\infty(M, \mathbb{R})$ is an $\mathbb{R}$-algebra with $(f \odot g)(p) := f(p)g(p)$.

(d) $$\Omega^*(M) := \bigoplus_{i=0}^{\infty} \Omega^i(M) = \bigoplus_{i=0}^{m} \Omega^i(M)$$

is an $\mathbb{R}$-algebra via

$$\omega \odot \tau := \omega \wedge \tau$$

"algebra of differential forms on M".

Def 3.43: Let A be an $\mathbb{R}$-algebra and let $A_i \leq A$ be subvector spaces $(i \in \mathbb{N}_0)$ such that

$A = \bigoplus_{i=0}^{\infty} A_i$. The $\mathbb{R}$-algebra A is called a <u>graded</u> $\mathbb{R}$-algebra with grading $\bigoplus_{i=0}^{\infty} A_i$ if

(GA1) $A = \bigoplus_{i=0}^{\infty} A_i$

(GA2) $\forall i, j \in \mathbb{N}_0 : \forall a \in A_i \, \forall b \in A_j :$

$$a \odot b \in A_{i+j}.$$

End of Lecture 31.03.2023

<u>Example 3.44:</u>

(a) $\mathbb{R}[X, Y]$ is a graded algebra with grading

$$\mathbb{R}[X, Y] = \bigoplus_{i=0}^{\infty} \mathbb{R}[X, Y]_i$$

$\mathbb{R}[X, Y]_i := \{ P \in \mathbb{R}[X, Y] \mid P = 0$ or P is a homogeneous polynomial of degree $i \}$

$$= \sum_{\substack{s,t \in \mathbb{N}_0 \\ s+t=i}} \mathbb{R} X^s Y^t$$