Proof of Theorem 2.16: (Sketch)

We just consider the case
$M = U \subseteq \mathbb{R}^m$ open and
$N = V \subseteq \mathbb{R}^n$ open

$C^r(U,V)$ is open $C^r_S(U,\mathbb{R}^n)$
(Exercise!) Thus it is enough
to consider $V = \mathbb{R}^n$.

Let $f \in C^r(U,\mathbb{R}^n)$ and $\mathcal{N}$
be standard open neighborhood
of f, i.e.

$$\mathcal{N} := \mathcal{N}(f, \underset{\underset{(K_i)_I}{\shortparallel}}{K}, \underset{\underset{(\varepsilon_i)_I}{\shortparallel}}{\varepsilon})$$

such that
- K_i is compact
- K is locally finite
- $\bigcup_I K_i = U$
- $\varepsilon_i > 0$

and $\mathcal{N}(f, K, \varepsilon) = \{h \in C^r(U, \mathbb{R}^n) \mid$

$\|h - f\|_{r, K_i} < \varepsilon_i \ \forall_{i \in I}\}$

(Why do those sets form a neighborhood basis w.r.t. strong topology? Exercise!)

To show $C^\infty(U, \mathbb{R}^n) \cap \mathcal{N} \neq \emptyset$.

Take a partition of unity $(\lambda_i)_{i \in I}$

s.t. $K_i \subseteq \{\underline{x} \in \mathbb{R}^m \mid \lambda_i(x) > 0\}$

$\ulcorner$ Take a locally finite $(U_i)_I$ open covering of U such that $K_i \subseteq U_i$.

Then take W_i open with $\overline{W_i}$ compact such that

$$K_i \subseteq W_i \subseteq \overline{W_i} \subseteq U_i$$

Now take a continuous

map $b_i : U_i \longrightarrow [0,1]$

$b_i|_{K_i} \equiv 1$, $b_i|_{U_i \setminus \overline{W_i}} \equiv 0$, $b_i > 0$ on W_i

Now convolute b_i with a θ_σ for a small σ to get $\tilde{\lambda}_i : U_i \to [0,1]$ such that

$\tilde{\lambda}_i \equiv 0$ near $Bd(U_i)$ and
$\tilde{\lambda}_i > 0$ on K_i.

Put $$\lambda_i := \frac{\tilde{\lambda}_i}{\sum_{i \in I} \tilde{\lambda}_i}.$$

$\lrcorner$

By Remark 2.22 (iii)

$\exists\, (g_i)_I$, $g_i \in C^\infty(U_i, \mathbb{R}^n)$

such that $\| g_i - f \|_{r,K} =: \varepsilon_i$

is small.

$g := \sum_{i \in I} \lambda_i g_i$ satisfies

$g \in \mathcal{N}(f, K, \varepsilon)$ for small enough $(\alpha_i)_I$ $\square$

<u>Convention 2.23:</u>

From now on, if we say "smooth manifold" we mean C^∞ manifold

Chapter III

De Rham Complex

Here all manifolds are smooth.

Def 3.1 (vector field)
Let M be a manifold. (a) A vector field on M is a map

$$X : M \longrightarrow TM \text{ such that}$$
for all $p \in M$ we have $X(p) \in T_pM$.

It is also called a section of the tangent bundle $\pi : TM \longrightarrow M$.

(b) Let X be a vector field on M. We call X smooth if X is a smooth map between the C^∞-manifolds M and TM.

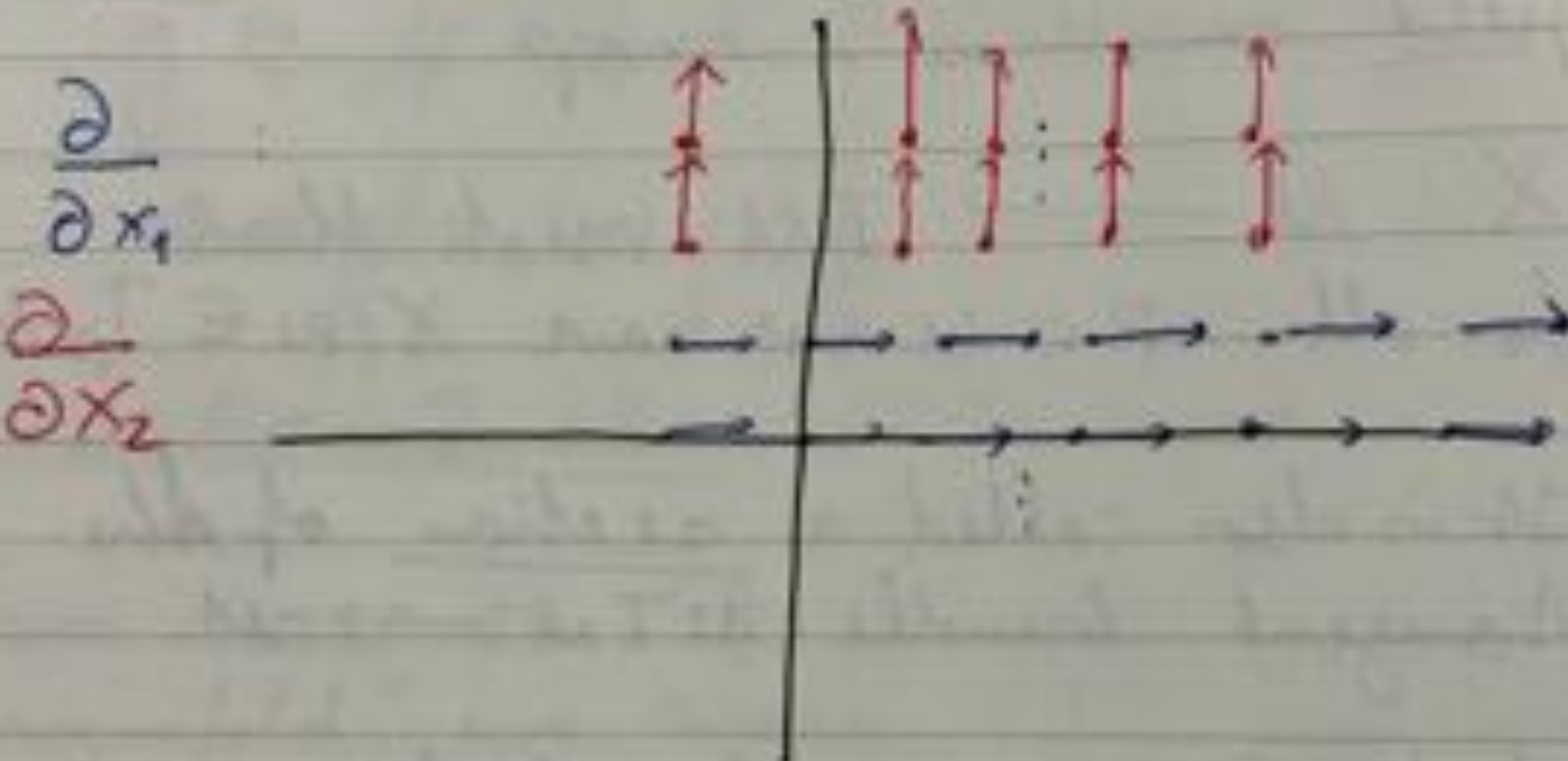
Example 3.2

(a) $M = \mathbb{R}^2$ Take the chart $(\varphi, \mathbb{R}^2) = (id_{\mathbb{R}^2}, \mathbb{R}^2)$, $\varphi = (x_1, x_2)$ "coordinates".

$$\frac{\partial}{\partial x_1}(p) = [p, id_{\mathbb{R}^2}, e_1], \frac{\partial}{\partial x_2}(p) = [p, id_{\mathbb{R}^2}, e_2]$$

are the derivations w.r.t. x_1 and x_2 at p.

Then $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ are vector fields ~~at~~ on $\mathbb{R}^2$



(b) Consider the ODE.

$y' = f(x,y) = |y|^{\frac{1}{2}}$ $\quad f \in C(\mathbb{R}^2; \mathbb{R})$

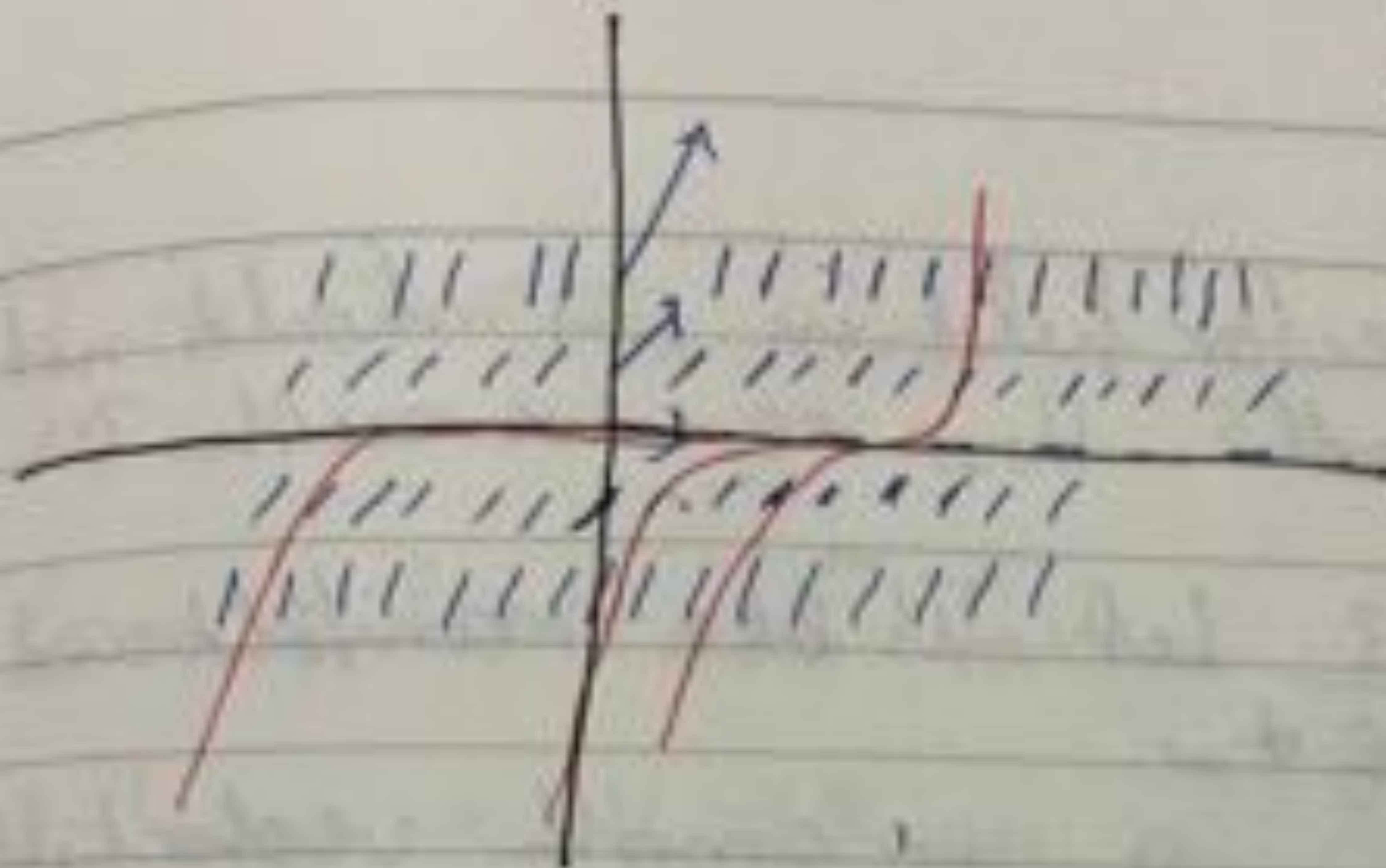
It gives a vector field

$X(p) = \frac{\partial}{\partial x_1}(p) + f(p) \frac{\partial}{\partial x_2}(p)$

$\cancel{X(x_1,x_2) = \frac{\partial}{\partial x_1}(x_1,x_2) + f(x_1,x_2)\frac{\partial}{\partial x_2}(x_1,x_2)}$

on $\mathbb{R}^2$.

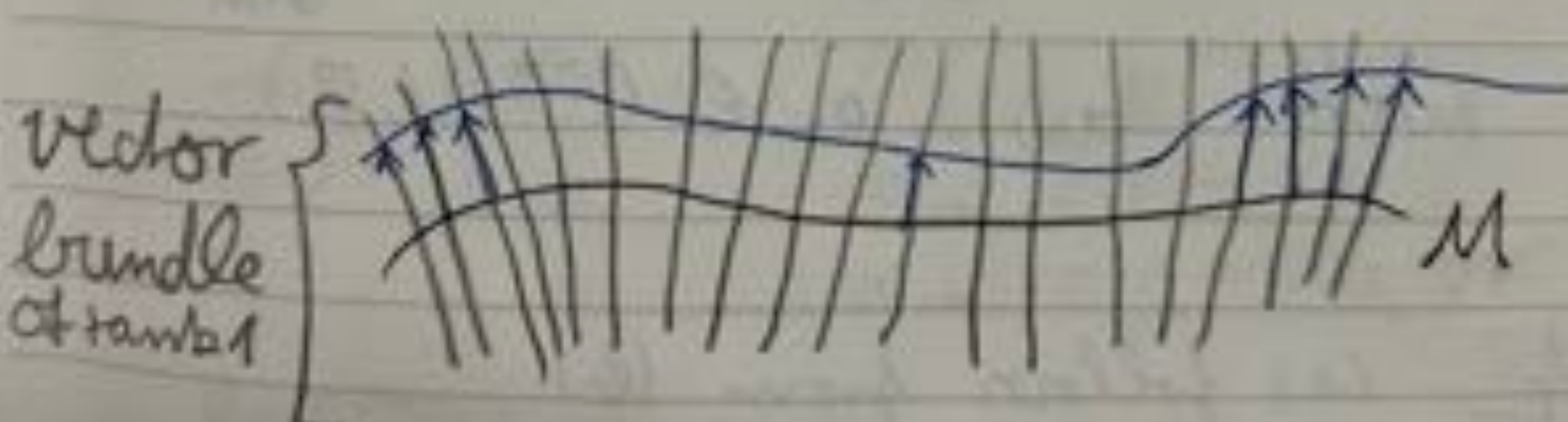
$$X = \frac{\partial}{\partial x_1} + \sqrt{|x_2|}\,\frac{\partial}{\partial x_2}$$



We can guess the solutions.

(c) Terminology "section:

Consider a vector bundle on a manifold:



We cut through the bundle "section".

(It is a bit more difficult to draw for a tangent bundle.)

The vector field in (b) should not be smooth. How can we see this?

<u>Lemma 3.3:</u> Let M^m be a manifold and $(\varphi, U) \in \mathcal{A}_\infty$.

(a) $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}$ are smooth vectorfields on U.

(b) A vector field X on U is smooth iff it has the form

$$X(p) = a_1(p)\frac{\partial}{\partial x_1}(p) + \dots + a_m(p)\frac{\partial}{\partial x_m}(p)$$

with $a_1, \dots, a_m \in C^\infty(U, \mathbb{R})$.

<u>Proof:</u> (a) follows from (b)

<u>Pf of (b)</u> TU has the global chart

$$\begin{aligned} TU &\xrightarrow{\Phi} \varphi(U) \times \mathbb{R}^m \\ [p, \varphi, v] &\longmapsto (\varphi(p), v) \end{aligned}$$

$\frac{\partial}{\partial x_1}(p) = [p, \varphi, e_1], \frac{\partial}{\partial x_2}(p), \dots, \frac{\partial}{\partial x_m}(p)$

form a basis of $T_p U$

$\Rightarrow \exists!$ maps $a_i : U \to \mathbb{R}, i=1,\dots,m$: $X(p) = \sum_{i=1}^{m} a_i(p) \frac{\partial}{\partial x_i}(p)$

Thus $X(p) = \sum a_i(p) [p, \varphi, e_i] = [p, \varphi, \sum a_i(p) e_i]$

and $\Phi \circ X = (\varphi, a_1, \dots, a_m)$

Thus $X \in C^\infty(U, TU)$

$\Leftrightarrow \Phi \circ X \in C^\infty(U, \mathbb{R}^m \times \mathbb{R}^m)$

$\Leftrightarrow \varphi, a_1, \dots, a_m \in C^\infty(U, \mathbb{R})$

$\Leftrightarrow a_1, \dots, a_m \in C^\infty(U, \mathbb{R})$ $\square$

$\uparrow$

$\varphi \in C^\infty(U, \mathbb{R})$

already

Def 3.4: Let M^m be a manifold.

(i) An (ordered) tuple $(X_1, \dots, X_m)$ of vectorfields on some open set $U \subseteq M$ is called a frame on U if

$\forall p \in U$: $(X_1(p), \dots, X_m(p))$ is an ordered basis of T_pU.

(ii) A local frame around p is a frame on some open neighborhood of p.

(iii) A frame on M is called a global frame.

Example 3.5: $(\varphi, U) \in \mathcal{A}_\infty$.

$\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\right)$ is a frame on U

$\left(f\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\right)$ — " —

$m \geq 2$: $\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\right)$ — " —

They all are smooth frames.

i.e. all the vector fields are smooth

Question 3.6 Is it possible to integrate a function on M, say $f \in C_c(M, \mathbb{R})$?

$\nwarrow$ compact support

$M = \mathbb{R}$, $f \in C_c(\mathbb{R}, \mathbb{R})$, e.g.

$$f(t) := \begin{cases} 1 - t^2, & t \in [-1, 1] \\ 0, & |t| \geq 1. \end{cases}$$

Chart $(\varphi, \mathbb{R})$: Put $I_\varphi(f) := \int_{-\infty}^{\infty} f(\varphi^{-1}(x))\,dx$.

For two charts $(\varphi, \mathbb{R})$, $(\psi, \mathbb{R})$ we have

$$I_\psi(f \cdot (D(\varphi \circ \psi^{-1}) \circ \psi)) = I_\psi\left(f \frac{\partial x}{\partial y}\right)$$

$$= \int_{-\infty}^{\infty} f(\psi^{-1}(y)) \frac{\partial x}{\partial y}(\psi^{-1}(y))\,dy$$

$$= \int_{\varphi(\psi^{-1}(-\infty))}^{\varphi(\psi^{-1}(\infty))} f(\varphi^{-1}(x))\,dx$$

$$= \begin{cases} \int_{-\infty}^{\infty} f(\varphi^{-1}(x))\,dx = I_\varphi(f), & D(\varphi\circ\psi^{-1}) > \\ \int_{-\infty}^{\infty} f(\varphi^{-1}(x))(-1)\,dx = -I_\varphi(f), & \text{if } D(\varphi\circ\psi^{-1}) < 0 \end{cases}$$

So instead of f we should integrate objects that look like

$f\,dx$ in x- coo. and

transform to $f\,dx = (D(\varphi\circ\psi^{-1})\circ\psi)\,dy$
$= f \dfrac{\partial x}{\partial y}\,dy$

in y- coo.

<u>And</u> the integral should only be taken w.r.t. charts φ and ψ such that $D(\varphi\circ\psi^{-1}) > 0$
(or for higher dimension

$$\det\big(\underbrace{D(\varphi\circ\psi^{-1})(\psi(P))}_{\shortparallel}\big) > 0$$

$$\det\left(\frac{\partial(x_1,\dots,x_m)}{\partial(y_1,\dots,y_m)}\right) \text{ (notation.)}$$

>0 We have to define those objects rigorously.

End of Lecture 21.03.23

Def 3.7: Let V be an $\mathbb{R}$-vector space and $k \in \mathbb{N}$.

An alternating k-form on V is a map

$$f : \underbrace{V \times \dots \times V}_{k} \longrightarrow \mathbb{R}$$

such that

(A1) f is $\mathbb{R}$-k-linear, i.e.

$$\forall_{i=1,\dots,k} \; \forall v_1, v_2, \dots, \hat{v}_i, \dots, v_k \in V$$

$$v \longmapsto f(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_k)$$

is $\mathbb{R}$-linear, and

(A2) $\forall v_1, \dots, v_k \in V$ linearly dependent:

$$f(v_1, \dots, v_k) = 0$$

Remark 3.8: Given (A1) then are equivalent

(i) (A2)

(ii) (A2') f is skew-symmetric, i.e. $\forall v_1, \dots, v_k \in V \; \forall_{\substack{i \neq j \\ 1 \leq i, j \leq k}}$

$$f(v_1,\dots,v_i,\dots,v_j,\dots,v_k)$$
$$= - f(v_1,\dots,v_j,\dots,v_i,\dots,v_k)$$

(A2'') $\forall v_1,\dots,v_k \in V \ \forall_{1 \le i \ne j \le k}$

$$\left(f(v_1,\dots,v_k) = 0 \text{ if } v_i = v_j \right)$$

Proof: (A2) $\Rightarrow$ (A2'') ✓

(A2'') $\Rightarrow$ (A2'): $(k=2)$

$$f(v_1+v_2, v_1+v_2) = 0$$
$$\| $$
$$f(v_1,v_2) + f(v_2,v_1)$$

$\Longrightarrow$ (A2').

(A2') $\Rightarrow$ (A2''): ~~Take a basis~~

$(k=2)$ $f(v,v) = -f(v,v)$

$$\Rightarrow 2 f(v,v) = 0 \underset{2 \in \mathbb{R}^\times}{\Rightarrow} f(v,v) = 0.$$

(A2'') $\Rightarrow$ (A2): Suppose $v_1,\dots,v_k$ are linearly dependent.

$$\Rightarrow \exists (\lambda_1,\dots,\lambda_k) \in \mathbb{R}^k \setminus \{(0)\} :$$
$$\sum_{i=1}^{k} \lambda_i v_i = 0$$

Wlog. $\lambda_k = -1$, i.e.

$$v_k = \sum_{i=1}^{k-1} \lambda_i v_i$$

$$\Rightarrow f(v_1,\dots,v_{k-1},v_k) = \sum_{i=1}^{k-1} f(v_1,\dots,v_{k-1},v_i)\,\lambda_i$$

$$= 0 \qquad \square$$

<u>Notation 39:</u> Let $W_1,\dots,W_k, E, V$ be $\mathbb{R}$ vs, $k \geq 1$.

- $\mathrm{Mult}(W_1, W_2, \dots, W_k, E)$
  $= \{ f : W_1 \times \dots \times W_k \to E \mid f$ is k-linear $\}$
- $\mathrm{Mult}^k(V, E) := \mathrm{Mult}(\underbrace{V, \dots, V}_{k}, E)$
- $\mathrm{Alt}^k(V, E) = \{ f \in \mathrm{Mult}^k(V, E) \mid$ f is alternating $\}$
- $\mathrm{Alt}^k(V) := \mathrm{Alt}^k(V, \mathbb{R})$.

Example 3.10:

(a) $\mathbb{R}^2$ as an $\mathbb{R}$-vector space

Which one is an ~~Thm~~ alternating 2-form?

$$f((x_1,y_1),(x_2,y_2)) := x_1 y_1 - x_2 y_2 = \det\begin{pmatrix} x_1 & y_2 \\ x_2 & y_1 \end{pmatrix}$$

or $$g((x_1,y_1),(x_2,y_2)) := x_1 y_2 - x_2 y_1 = \det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \,?$$

(b) $\mathrm{Alt}^1(V) = L(V, \mathbb{R})$

(c) If $\dim_{\mathbb{R}} V = m$, then $\dim_{\mathbb{R}} \mathrm{Alt}^m(V) = 1$

because for $a \in \mathrm{Alt}^m(V)$ we have for a basis $v_1, \dots, v_m$ of V and $w_1, \dots, w_m \in V$:

$$a(w_1, \dots, w_m) = a\Big(\sum_i v_i^*(w_1) v_i, \dots, \sum_i v_i^*(w_m) v_i$$

$$= \sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn}(\sigma)\, v_1^*(w_{\sigma(1)})\, v_2^*(w_{\sigma(2)}) \cdots v_m^*(w_{\sigma(m)})$$

$$\cdot\, a(v_1, v_2, v_3, \dots, v_m)$$

$$=: \quad a(v_1, \dots, v_m)\, \underbrace{(v_1^* \wedge \dots \wedge v_m^*)}_{\substack{\text{wedge product} \\ \text{of } v_1^*, \dots, v_m^*}}(w_1, \dots, w_m)$$

So a is determined by its value on $(v_1, \dots, v_m)$.

$\Rightarrow \dim_{\mathbb{R}} \mathrm{Alt}^m(V) \leq 1.$

It is 1, because $v_1^* \wedge \dots \wedge v_m^* \in \mathrm{Alt}^m(V)$ and $(v_1^* \wedge \dots \wedge v_m^*)(v_1, \dots, v_m) = 1$.

<u>Ex.</u> ~~xxxx~~ $V = \mathbb{R}^m$:

$$(e_1^* \wedge e_2^* \wedge \dots \wedge e_m^*)(w_1, \dots, w_m)$$

$$= \det \begin{pmatrix} e_1^*(w_1), & \dots, & e_1^*(w_m) \\ \vdots & & \vdots \\ e_m^*(w_1), & \dots, & e_m^*(w_m) \end{pmatrix}$$

(d) $V = \mathbb{R}^3$ $\quad Alt^2(\mathbb{R}^3)$

$$\dim_{\mathbb{R}} Alt^2(\mathbb{R}^3) = 3.$$

$a \in Alt^2(\mathbb{R}^3)$.

$$a(\overbrace{\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3}^{w_1}, \overbrace{\mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3}^{w_2})$$

$$= (\lambda_1 \mu_2 - \lambda_2 \mu_1)\, a(e_1, e_2)$$
$$+ (\lambda_1 \mu_3 - \lambda_3 \mu_1)\, a(e_1, e_3)$$
$$+ (\lambda_2 \mu_3 - \lambda_3 \mu_2)\, a(e_2, e_3)$$

$$= a(e_1, e_2) \cdot e_1^* \wedge e_2^* (w_1, w_2)$$
$$+ a(e_1, e_3) \cdot e_1^* \wedge e_3^* (w_1, w_2)$$
$$+ a(e_2, e_3) \cdot e_2^* \wedge e_3^* (w_1, w_2)$$

Here $(e_1^* \wedge e_2^*)(w_1, w_2) := e_1^*(w_1) e_2^*(w_2) - e_1^*(w_2) e_2^*(w_1)$

<u>Définition 3.11:</u> (wedge product)

$$\wedge : Alt^k(V) \times Alt^l(V) \longrightarrow Alt^{k+l}(V)$$

$$(a \wedge b)(v_1, \dots, v_{k+l}) :=$$

$$\frac{1}{k!\,l!}\sum_{\sigma\in S_{k+l}} \operatorname{sgn}(\sigma)\, a(v_{\sigma(1)},\dots,v_{\sigma(k)})\, b(v_{\sigma(k+1)},\dots,v_{\sigma(k+l)})$$

"a wedge b"

<u>Lemma 3.12:</u> Let V be an $\mathbb{R}$-vector space and $k, l, m \in \mathbb{N}$.

(a) $\forall a \in \mathrm{Alt}^k(V)\ \forall b \in \mathrm{Alt}^l(V)\ \forall c \in \mathrm{Alt}^m(V)$:
$(a \wedge b)\wedge c = a \wedge (b \wedge c)$

(b) Suppose V is finite dimensional and $e_1, \dots, e_n$ is a basis of V with dual basis $e_1^*, \dots, e_n^*$.

Then $\{ e_{i_1}^* \wedge e_{i_2}^* \wedge \dots \wedge e_{i_k}^* \mid 1 \le i_1 < i_2 < \dots < i_k \le n \}$
is a basis of $\mathrm{Alt}^k(V)$.

($e_{i_1}^* \wedge \dots \wedge e_{i_k}^*$ is the alternating form a which satisfies

$$a(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1, & i_1 = j_1 \wedge i_2 = j_2 \wedge \dots \wedge i_k = j_k \\ 0, & i_1 \ne j_1 \vee i_2 \ne j_2 \vee \dots \vee i_k \ne j_k \end{cases}$$

for $1 \le j_1 < j_2 < \dots < j_k \le n$.)

Proof: (a) $\frac{1}{(k+l)!\,m!} \cdot \frac{1}{k!\,l!} \sum_{\rho \in \mathfrak{S}_{k+l+m}} \sum_{\sigma \in \mathfrak{S}_{k+l}} \operatorname{sgn}(\rho) \operatorname{sgn}(\sigma) \cdot$

$\cdot a(v_{\sigma(\rho(1))}, \dots, v_{\sigma(\rho(k))}) b(v_{\sigma(\rho(k+1))}, \dots, v_{\sigma(\rho(k+l))})$

$\cdot c(v_{\rho(k+l+1)}, \dots, v_{\rho(k+l+m)})$

$= \frac{1}{k!\,l!\,m!\,(k+l)!} \sum_{\sigma} \operatorname{sgn}(\sigma) \sum_{\rho} \operatorname{sgn}(\rho)\, a(\dots) b(\dots) c(\dots)$

$= (*)$.

Define $d(w_1, \dots, w_{k+l+m})$ as

$\sum_{\rho \in \mathfrak{S}_{k+l+m}} \operatorname{sgn}(\rho)\, a(v_{\rho(1)}, \dots, v_{\rho(k)}) b(v_{\rho(k+1)}, \dots, v_{\rho(k+l)})$

$c(v_{\rho(k+l+1)}, \dots, v_{\rho(k+l+m)})$

Then $d \in \mathrm{Alt}^{k+l+m}(V)$.

Now consider $\mathfrak{S}_{k+l} \overset{\iota}{\hookrightarrow} \mathfrak{S}_{k+l+m}$

$\sigma \longmapsto \iota(\sigma)$

$$\iota(\sigma)(z) := \begin{cases} \sigma(z), & z \in \{1, \dots, k+\ell\} \\ z, & z \in \{k+\ell+1, \dots, m\}. \end{cases}$$

Then $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\iota(\sigma))$ because ι respects transpositions.

Then $(*) = \frac{1}{k!\,\ell!\,m!\,(k+\ell)!} \sum_{\sigma \in \mathfrak{S}_{k+\ell}} d(v_1, \dots, v_{k+\ell+m})$.

$$= \frac{1}{k!\,\ell!\,m!} \, d(v_1, \dots, v_{k+\ell+m})$$

$$\underset{\substack{\uparrow \\ \text{similar}}}{=} \quad a \wedge (b \wedge c)(v_1, \dots, v_{k+\ell+m}).$$

(b) Exercise. $\square$

<u>Example 3.13:</u> Take $V = \mathbb{R}^n$.

$(e_1^* \wedge \dots \wedge e_n^*)(v_1, \dots, v_n)$ is the signed volume of the parallelotop spanned by $v_1, \dots, v_n$.

The sign is negative if

$\det(v_1, \dots, v_n) < 0$,

columns

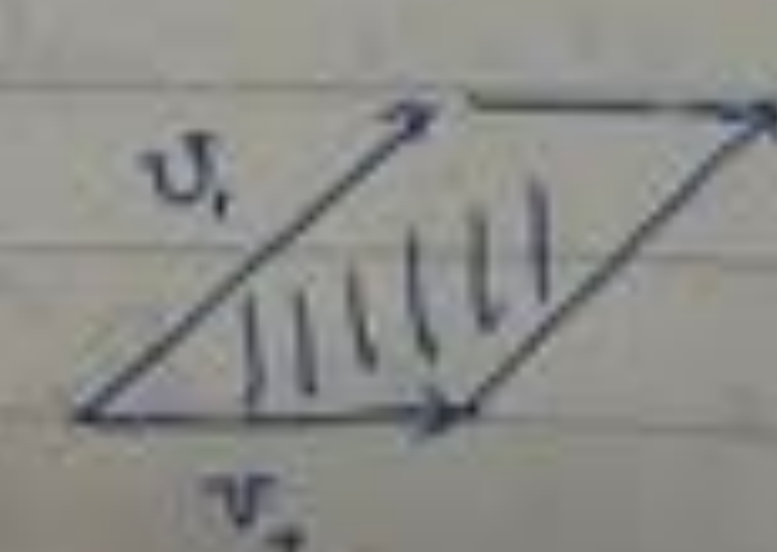
i.e. the base change matrix A from $(e_1, \dots, e_n)$ to $(v_1, \dots, v_n)$ has negative determinant

$$(v_1, \dots, v_n) A = (e_1, \dots, e_n)$$

(If we have the coordinates w.r.t. $e_1, \dots, e_n$ we get the coo. w.r.t. $v_1, \dots, v_n$: $w = \sum \lambda_i e_i$

$\Rightarrow A \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$ coo w.r.t. $(v_1, \dots, v_n)$

<u>n=2:</u> $v_1 = \begin{pmatrix} 1 \\ a \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $a > 0$



area $= a$
signed area $= -a$

$$(e_1^* \wedge e_2^*)(v_1, v_2) = \det \begin{bmatrix} e_1^*(v_1) & e_2^*(v_1) \\ e_1^*(v_2) & e_2^*(v_2) \end{bmatrix}$$

$$= \det\begin{pmatrix} 1 & a \\ 1 & 0 \end{pmatrix} = -a$$

Def 3.14: (Tensor product)

Let $V_1, \dots, V_k$ be real vector spaces of finite dimension.
A pair $(E, \otimes)$ consisting of
(a) a real vector space E of dimension
$\dim V_1 \cdot \ldots \cdot \dim V_k$, and
(b) a k-linear map
$\otimes : V_1 \times \dots \times V_k \longrightarrow E$
such that $\operatorname{im}(\otimes)$ spans E,
is called a tensor product of $V_1, V_2, \dots, V_k$.

(Definition for infinite dimensional spaces is given in the problem sheet.)

Example 3.15: (i) $M(n,m)(\mathbb{R}) = \{ A \mid$
A a matrix with real entries and n-rows and m columns $\}$ with

$1/21$

$$\otimes : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow M(n,m)(\mathbb{R})$$
$$(v, w) \longmapsto v \cdot w^t$$

is a tensor product of $\mathbb{R}^n$ with $\mathbb{R}^m$.

$$A \in M(n,m)(\mathbb{R})$$
$$= \sum_{i,j} a_{ij} \, e_i \cdot e_j^t$$

(ii) ~~$\mathbb{R}^n \otimes \mathbb{R}^m \otimes \mathbb{R}^\ell$~~ Notation: $V^* := L(V, \mathbb{R})$ "dual vector spac

$\mathrm{Mult}(\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^\ell, \mathbb{R})$ together with

$$\otimes_M : (\mathbb{R}^n)^* \times (\mathbb{R}^m)^* \times (\mathbb{R}^\ell)^* \longrightarrow \mathrm{Mult}(\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}$$

$$(f, g, h) \longmapsto \left((v_1, v_2, v_3) \mapsto f(v_1) g(v_2) h\right.$$

is a tensor product of $(\mathbb{R}^n)^*, (\mathbb{R}^m)^*, (\mathbb{R}$

(iii) In (ii) we have

$$\Phi_n : (\mathbb{R}^n)^* \xrightarrow{\sim} \mathbb{R}^n \text{ via}$$
$$f \longmapsto \begin{pmatrix} f(e_1) \\ f(e_n) \end{pmatrix}$$

Thus with $\mathrm{Mult}(\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^\ell, \mathbb{R})$

$$\otimes_{iii} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell \longrightarrow \mathrm{Mult}(\dots)$$

$$\Theta_{iii} = \left(\Phi_n^{-1} \times \Phi_m^{-1} \times \Phi_\ell^{-1}\right) \circ \otimes_{ii}$$

is a tensor product of $\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^\ell$.

We can consider $\mathrm{Mult}(\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^\ell, \mathbb{R})$ as a set of 3-dimensional matrices (tensors of ~~3~~ degree 3)

$$\mathrm{Mult}(\dots) \xrightarrow{\sim} M_{(n,m,\ell)}(\mathbb{R})$$

$$b \longmapsto \left(b(e_i, e_j, e_k)\right)_{\substack{1 \le i \le n \\ 1 \le j \le m \\ 1 \le k \le \ell}}$$

Ex: $(n, m, \ell) = (2, 2, 3)$

$= A \qquad a_{112} = 3.$

Think how to multi-
ply A with vectors.

Prop 3.16: (Uniqueness of tensor product)

Let $(E, \otimes)$ and $(E', \otimes')$ be tensor products of $V_1, \dots, V_k$. Then there are linear bijections $f_i \in L(V_i, V_i)$, $i = 1, \dots, k$, $g \in L(E, E')$ such that the diagramm

$$\begin{tikzcd} V_1 \times \dots \times V_k \arrow[r, "f_1 \times \dots \times f_k"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\circlearrowleft"] & V_1 \times \dots \times V_k \arrow[d, "\otimes'"] \\ E \arrow[r, "g"] & E' \end{tikzcd}$$

commutes.
In fact we can take $f_1 = id_{V_1}$, $f_2 = id_{V_2}, \dots, f_k = id_{V_k}$.

Proof: Let $\{v_j^{(i)} \mid j = 1, \dots, d_i\}$ be a basis of V_i

Then $\{ v_{j_1}^{(1)} \otimes v_{j_2}^{(2)} \otimes \ldots \otimes v_{j_k}^{(k)} \mid$

$1 \leq j_1 \leq d_1, 1 \leq j_2 \leq d_2, \ldots, 1 \leq j_k \leq d_k \}$
is a basis of E.
We define g via
$g(v_{j_1}^{(1)} \otimes \ldots \otimes v_{j_k}^{(k)}) := v_{j_1}^{(1)} \otimes' \ldots \otimes' v_{j_k}^{(k)}$

End of Lecture 24.03.23 $\square$

Notation 3.17: $V_1 \otimes \ldots \otimes V_k$ for

the tensor product of $V_1, \ldots, V_k$.

The elements of $V_1 \otimes \ldots \otimes V_k$ are called tensors.
The elements of — " — of the form $v_1 \otimes v_2 \otimes \ldots \otimes v_k$ are called elementary tensors.

Remark 3.18: (a) A tensor does not need to be elementary!

$V = \mathbb{R}^2$ $\quad e_1 \otimes e_2$ is elementary
$e_1 \otimes e_2 + e_2 \otimes e_1$ not