DIFFERENTIAL TOPOLOGY
PROBLEM SHEET 7

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Problem 1 (20, tensor product)**.** Let V_1, V_2 be real vector spaces. A pair $(E, \otimes)$ is called *tensor product* of $V_!$ and V_2 if E is an $\mathbb{R}$-vector space and $\otimes$ is a biliniear map from $V_1 \times V_2$ to E such that the following holds: For every biliniear map $b : V_1 \times V_2 \to W$ there is a unique linear map $g : E \to W$ such that $b = g \circ \otimes$ (*Universal property of the tensor product*). Prove:

(i) Suppose there are two tensor products $(E_1, \otimes_1)$, $(E_2, \otimes_2)$ for V_1, V_2 then there is a linear isomorphism $f : E_1 \to E_2$ such that $f \circ \otimes_1 = \otimes_2$.
(ii) There exists a tensor product for V_1 and V_2.

Problem 2 (10, exterior power)**.** Let V be a finite dimensional real vector space. Prove that $\Lambda^k(V^*)$ is isomorphic to $\mathrm{Alt}^k(V)$ for all positive integers k.

Problem 3 (10 points, associativity of the tensor product)**.** Let V_1, V_2, V_3 be real vector spaces. Prove that there is a natural isomorphism from $(V_1 \otimes V_2) \otimes V_3$ to $V_1 \otimes (V_2 \otimes V_3)$.

Problem 4 (10, k-tensor product)**.** Let $k \geqslant 3$ be a positive integer. Let $V_1, \ldots, V_k$ be real vector-spaces and put

$$E := (\ldots(V_1 \otimes V_2) \otimes V_3) \otimes \cdots \otimes V_{k-1}) \otimes V_k.$$

Formulate and prove the universal property for the *k-tensor product* E.

Date: Please hand in before the lecture by **31.03.2023**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference.