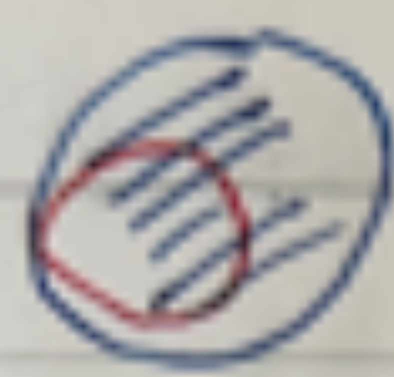
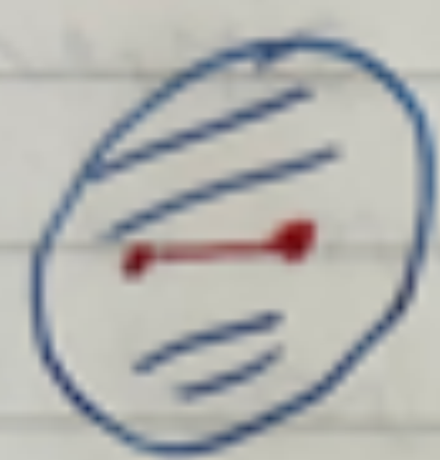
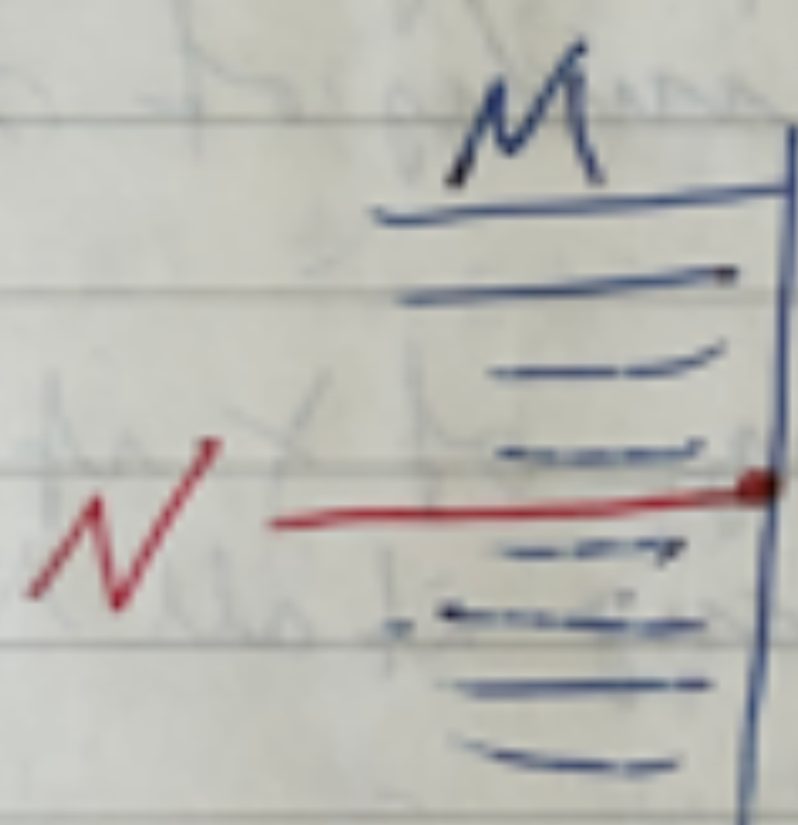


not neat, because $\partial N = \emptyset \subsetneq N \cap \partial M$.



not neat: $\partial N \supsetneq \emptyset = N \cap \partial M$.

(b)



N is neat.

Def 1.60: An C^r-embedding $f: N^n \to M^m$ is called neat if $f(N)$ is a neat C^r-submanifold of M.

Thm 1.61: Let M^n be a compact C^r-manifold, $r \geq 1$. Then there exists a neat C^r-embedding of M^n into $\mathbb{R}^{2n} \times \mathbb{R}^{\geq 0}$.

Chapter II

Approximation

We first recall the notions of covers and locally finiteness.

Def 2.1: Let X be a topological space and $\mathcal{A} = (A_i)_{i \in I}$ be a family of subsets of X.

(a) We call $\mathcal{A}$ a covering of X if $\bigcup_{i \in I} A_i = X$ (open covering if all A_i are open)

(b) We call $\mathcal{A}$ locally finite if $\forall x \in X$ $\exists U \subseteq X$ open nbhd of x: $|\{i \in I \mid A_i \cap U \neq \emptyset\}|$ is finite

(c) Let $\mathcal{A} = (A_i)_I$ and $\mathcal{B} = (B_j)_J$ be open coverings of X. We call $\mathcal{B}$

(i) a refinement of $\mathcal{A}$ if $\forall j \in J\ \exists i \in I$ $B_j \subseteq A_i$

(ii) a shrinking of $\mathcal{A}$ if ~~$\forall j \in J \exists i \in I$~~ $I = J$ and $\overline{B_i} \subseteq A_i$ for all $i \in I$.

Def 2.2 (Partition of unity)

Let $(U_i)_I =: \mathcal{A}$ be an open cover of a topological space X. A family $\Lambda := (\lambda_i)_I$ of continuous maps $\lambda_i : X \longrightarrow [0,1]$ is called a partition of unity subordinate to $\mathcal{A}$ if

(U1) $\operatorname{supp} \lambda_i \subseteq U_i \quad \forall i \in I$

(U2) $(\operatorname{supp}(\lambda_i))_I$ is locally finite

(U3) $\forall x \in X: \quad \sum_I \lambda_i(x) = 1.$

Remark 2.3: Given a p.o.u. Λ subordinate to $\mathcal{A}$ we obtain a locally finite cover of X: $(\operatorname{int}(\operatorname{supp} \lambda_i))_I$.

Proof: • cover: $x \in X \Rightarrow \sum_{i \in I} \lambda_i(x) = 1$
$\Rightarrow \exists i_0 \in I : \lambda_{i_0}(x) > 0$
$\Rightarrow \quad x \in \lambda_{i_0}^{-1}(]0, \infty[) \subseteq \operatorname{int}(\operatorname{supp} \lambda_{i_0})$
• Locally finite: It follows from (U2) □

Ex. 2.4: (a) $X = \mathbb{R}$. $\mathcal{A} = \{\mathbb{R}\}$. $\Lambda = \{\mathbb{1}_{\mathbb{R}}\}$

(b) $X = S^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$
$U_1 = S^1 \setminus \{-1\}$, $U_2 = S^1 \setminus \{1\}$
$\mathcal{A} = \{U_1, U_2\}$

Date

$$\tilde{\lambda}_2(e^{i\theta}) := \begin{cases} e^{-\frac{1}{(\theta-\frac{\pi}{4})(\frac{3\pi}{4}-\theta)}}, & \theta \in]\frac{\pi}{4}, \frac{3}{4}\pi[\\ 0, & \theta \in [-\frac{\pi}{4}, \frac{\pi}{4}] \end{cases}$$

$$\tilde{\lambda}_1(e^{i\theta}) := \begin{cases} e^{-\frac{1}{(\frac{3\pi}{4}-\theta)(\theta+\frac{3\pi}{4})}}, & \theta \in]-\frac{3}{4}\pi, \frac{3}{4}\pi[\\ 0, & \theta \in [-\frac{5}{4}\pi, -\frac{3}{4}\pi] \end{cases}$$

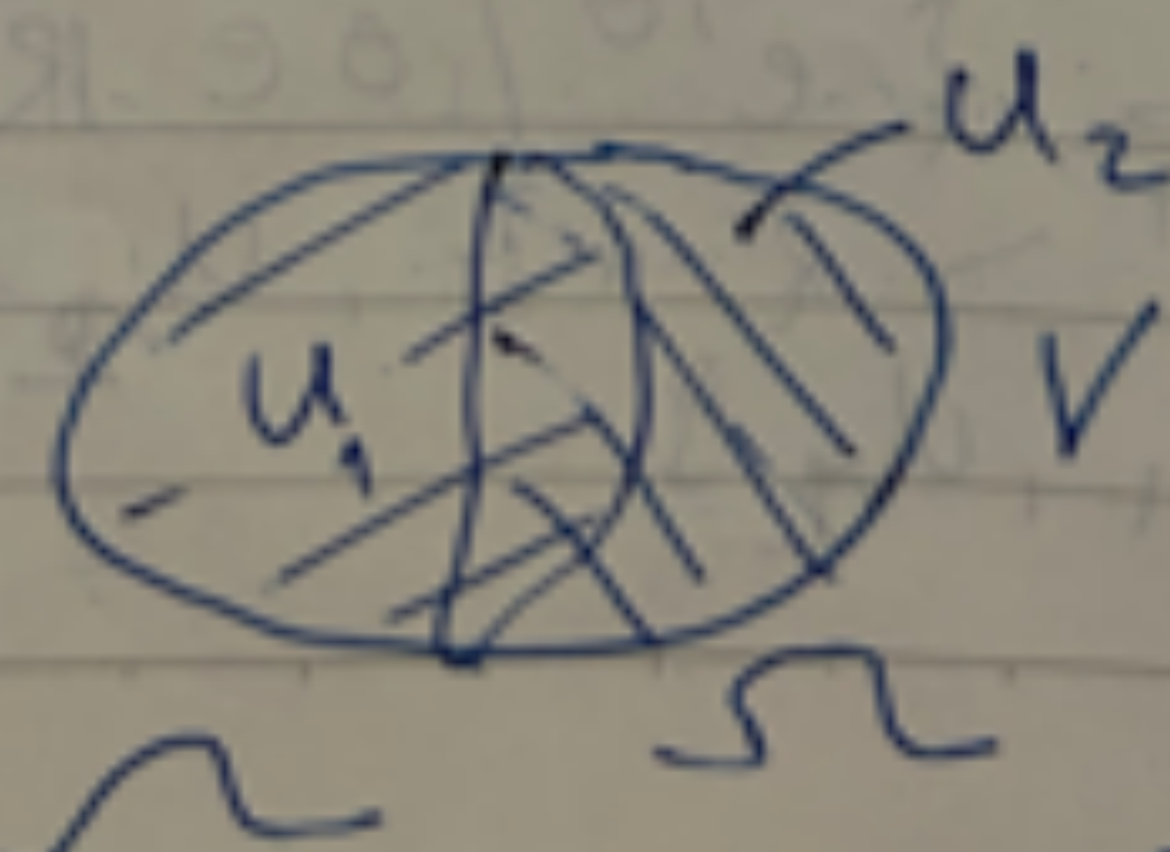
We define $\lambda_i(x) := \dfrac{\hat{\lambda}_i(x)}{\tilde{\lambda}_1(x) + \tilde{\lambda}_2(x)}$, $x \in S^1$.

Then $\operatorname{supp} \lambda_i \subseteq U_i$ and

$$\lambda_1 + \lambda_2 \equiv 1.$$

<u>Lemma 2.5:</u> Let $\mathcal{B}$ and $\mathcal{A}$ be open covers of X such that $\mathcal{A}$ refines $\mathcal{B}$. Then $\mathcal{B}$ has a subordinate p.o.u if $\mathcal{A}$ has one.

<u>Proof:</u> Idea



Consider $\lambda_1 + \lambda_2$ on V.

Let $(\lambda_i)_I$ be a P.o.U. subordinate to $\mathcal{A}$. We choose a map $f: I \to J$ such that

$$U_i \subseteq V_{f(i)} \text{ for all } i \in I$$

Put $\mu_j := \sum_{i \in f^{-1}(j)} \lambda_i$

(U1) $\operatorname{supp}(\mu_j) \subseteq \bigcup_{i \in f^{-1}(j)} \operatorname{supp}(\lambda_i)$ (Exercise)

$$\subseteq \bigcup_{i \in f^{-1}(j)} U_i \subseteq V_j$$

(U2) We ~~s~~show that $(\operatorname{supp}(\mu_j))_{j \in J}$ is locally finite

$x \in X \Rightarrow \exists\, U$ open nbhd of x: The set $S = \{i \in I \mid U \cap \operatorname{supp} \lambda_i \neq \emptyset\}$ is finite

Take $j \in J$ s.t. $(\operatorname{supp} \mu_j) \cap U \neq \emptyset$

$\Rightarrow$ (by (U1)) $\exists\, i \in I : f(i) = j$ and $(\operatorname{supp}(\lambda_i)) \cap U \neq \emptyset$.

$\Rightarrow \quad j \in f(S)$.

$f(S)$ is finite $\quad \square$ (U2)

(U3) $x \in X$: $\displaystyle\sum_{j \in J} \mu_j(x) = \sum_j \sum_{i \in f^{-1}(j)} \lambda_i(x)$

$$= \sum_{i \in I} \lambda_i(x) = 1. \qquad \square$$

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<u>Theorem 2.6:</u> Let M be a C^r manifold with $\partial M = \emptyset$. Then every open cover of M has subordinate C^r-partition of unity (meaning: consisting of C^r-maps λ_j.)

<u>Proof:</u> Let $\mathcal{A} = (A_i)_I$ be an open covering of M. Then we find an open refinement $\mathcal{B} = (B_j)$ with the following properties

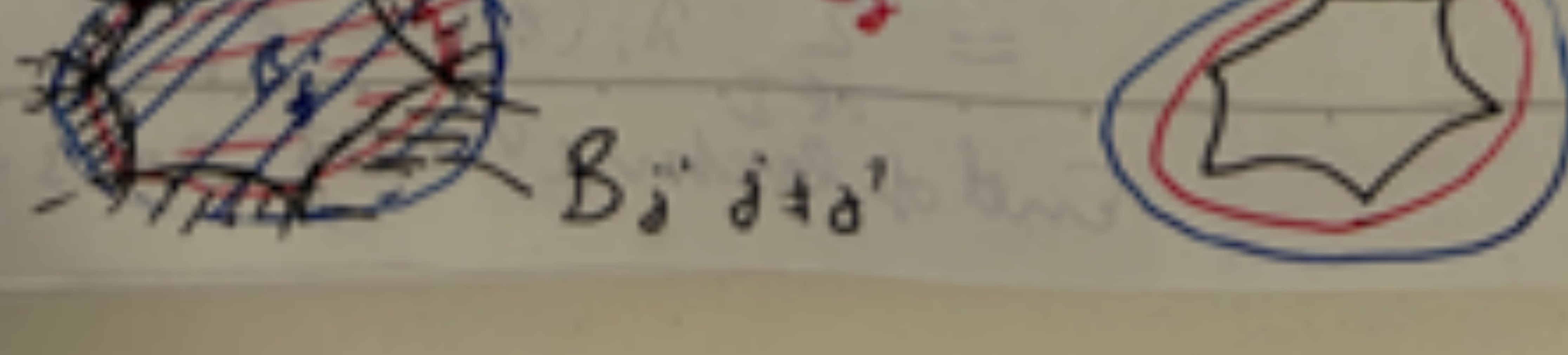
(1) $\overline{B}_j$ is compact.

(2) $\exists$ chart (φ_j, U_j) such that $\overline{B}_j \subseteq U_j$

(3) $\mathcal{B}$ is locally finite

To see this: Take at first a refinement $\mathcal{C}$ satisfying (1) and (2) and then a locally finite refinement of $\mathcal{C}$ to obtain $\mathcal{B}$. The last step uses the paracompactness. (Convention 1.10.)

We now take an open shrinking $\mathcal{C}$ of $\mathcal{B}$. (Why is this possible?)



Recall: $\mathcal{C} = (C_j)_j$ s.t. $\overline{C_j} \subseteq B_j$

Cover now $\varphi_j(\overline{C_j})$ with finitely many balls $K_t^{(j)}$, $t \in T^{(j)}$ such that $\bigcup_{t \in T^{(j)}} \overline{K}_t^{(j)} \subseteq \varphi_j(B_j)$.

Take C^∞ maps $\lambda_t^{(j)} = \begin{cases} >0, & \text{on } K_t^{(j)} \\ =0, & \text{else on } \mathbb{R}^n \end{cases}$

Then put

$$\lambda^{(j)} := \sum_{t \in T^{(j)}} \lambda_t^{(j)} : \mathbb{R}^n \to [0, \infty[.$$

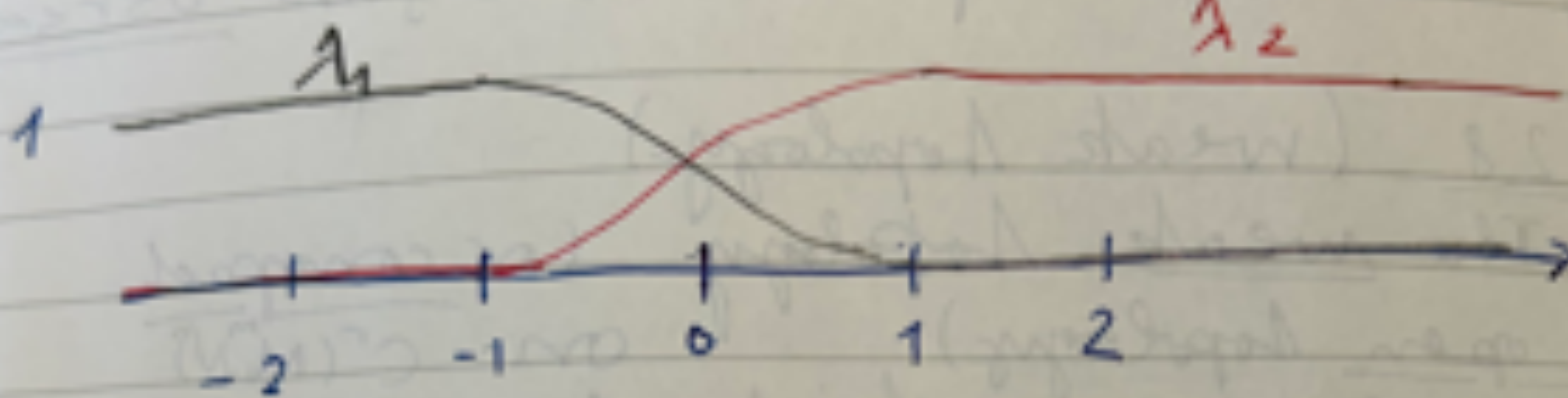
and $\mu^{(j)} = \begin{cases} \lambda^{(j)} \circ \varphi_j, & \text{on } B_j \\ 0, & \text{on } M \setminus B_j \end{cases}$

and $\nu^{(j)} := \dfrac{\mu^{(j)}}{\sum_{j' \in J} \mu^{(j')}}$.

$\square$

Example 2.7: (how to use p.o.u)

(a) $M = \mathbb{R}$, $U_1 =]-\infty, 2[$, $U_2 =]-2, \infty[$



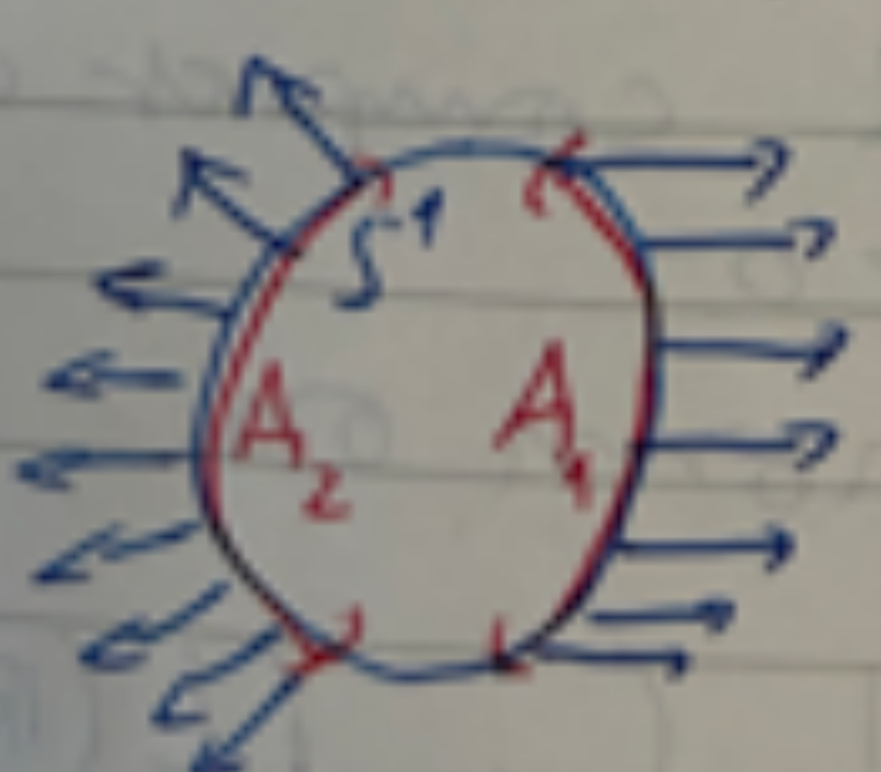
$$\lambda_1 + \lambda_2 = 1, \ supp(\lambda_i) \subseteq U_i$$

Say we are given $f_i \in C^\infty(U_i, \mathbb{R})$

How to find $f \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $f|_{]-\infty,-1]} = f_1$ and $f|_{[1,\infty[} = f_2$?

$$f = \lambda_1 f_1 + \lambda_2 f_2.$$

(b)



$A_1 = \{ e^{i\theta} \mid \theta \in]-\frac{3\pi}{8}, \frac{3\pi}{8}[\}$

$A_2 = \{ e^{i\theta} \mid \theta \in]\frac{5\pi}{8}, \frac{11\pi}{8}[\}$

$f_1 : A_1 \to \mathbb{R}^2 \quad f_1(e^{i\theta}) = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$f_2 : A_2 \to \mathbb{R}^2 \quad f_2(e^{i\theta}) = e^{i\theta} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

Find a smooth $f: S^1 \longrightarrow \mathbb{R}^2$
such that $f|_{A_i} = f_i$, $i = 1, 2$.

We want to approximate a map $f \in C^r(M,N)$ by a "better" map. Assume first $0 \leq r < \infty$

Def 2.8 (weak topology)
The weak topology (or compact open topology) on $C^r(M,N)$ is the coarsest topology containing the following sets:

$$\mathcal{N}^r(f, (\varphi, U), (\psi, V), K, \varepsilon)$$
$$:= \{ g \in C^r(M,N) \mid g(K) \subseteq V \text{ and }$$
$$\| D^k(\psi f \varphi^{-1})(\underline{x}) - D^k(\psi g \varphi^{-1})(\underline{x}) \|_k < \varepsilon$$
$$\forall \underline{x} \in \varphi(K) \; \forall k = 0, \ldots, r \}$$

For $f \in C^r(M,N)$, $(\varphi, U) \in \alpha_r$, $(\psi, V) \in \beta_r$ such that $K \subseteq U$ compact and $f(K) \subseteq V$, and $\varepsilon > 0$.
Here $\| \cdot \|_k$ is a norm on

$$\underbrace{L(\mathbb{R}^m, L(\mathbb{R}^m, L(\ldots}_{k\text{-times}} (L(\mathbb{R}^m, \mathbb{R}^n)) \ldots)$$

For $k=0$ we take a norm on $\mathbb{R}^n$.

<u>Example 2.9</u>: (a) For the norm ~~on~~ $\|\cdot\|_k$ we could take:

$$(F \in L(\mathbb{R}^m, \dots, L(\mathbb{R}^m, \mathbb{R}^n)\dots))$$

$$\|F\|_k := \sum_{\substack{(i_1,\dots,i_k) \\ \in \{1,\dots,m\}^k}} |F(e_{i_1})(e_{i_2})(e_{i_3})\dots(e_{i_k})|$$

For example for $f \in C^2(\mathbb{R}^m, \mathbb{R})$ we would get

$$\|D^0(f)(\underline{x})\|_0 = |f(\underline{x})|$$

$$\|D^1(f)(\underline{x})\|_1 = \sum_{i=1}^m \left|\frac{\partial f}{\partial x_i}(\underline{x})\right|$$

$$\|D^2(f)(\underline{x})\|_2 = \sum_{i=1}^m \sum_{j=1}^m \left|\frac{\partial^2 f}{\partial x_i \partial x_j}(\underline{x})\right|$$

(b) $M = \mathbb{R}$. $f \in C^2(\mathbb{R}, \mathbb{R})$

$f(x) = x^2$.

$g(x) = \begin{cases} x^2 + \frac{1}{1000} x^3 \end{cases}$

$K := [-1,1]$, $(\varphi, U) = (\psi, V) = (id, \mathbb{R})$

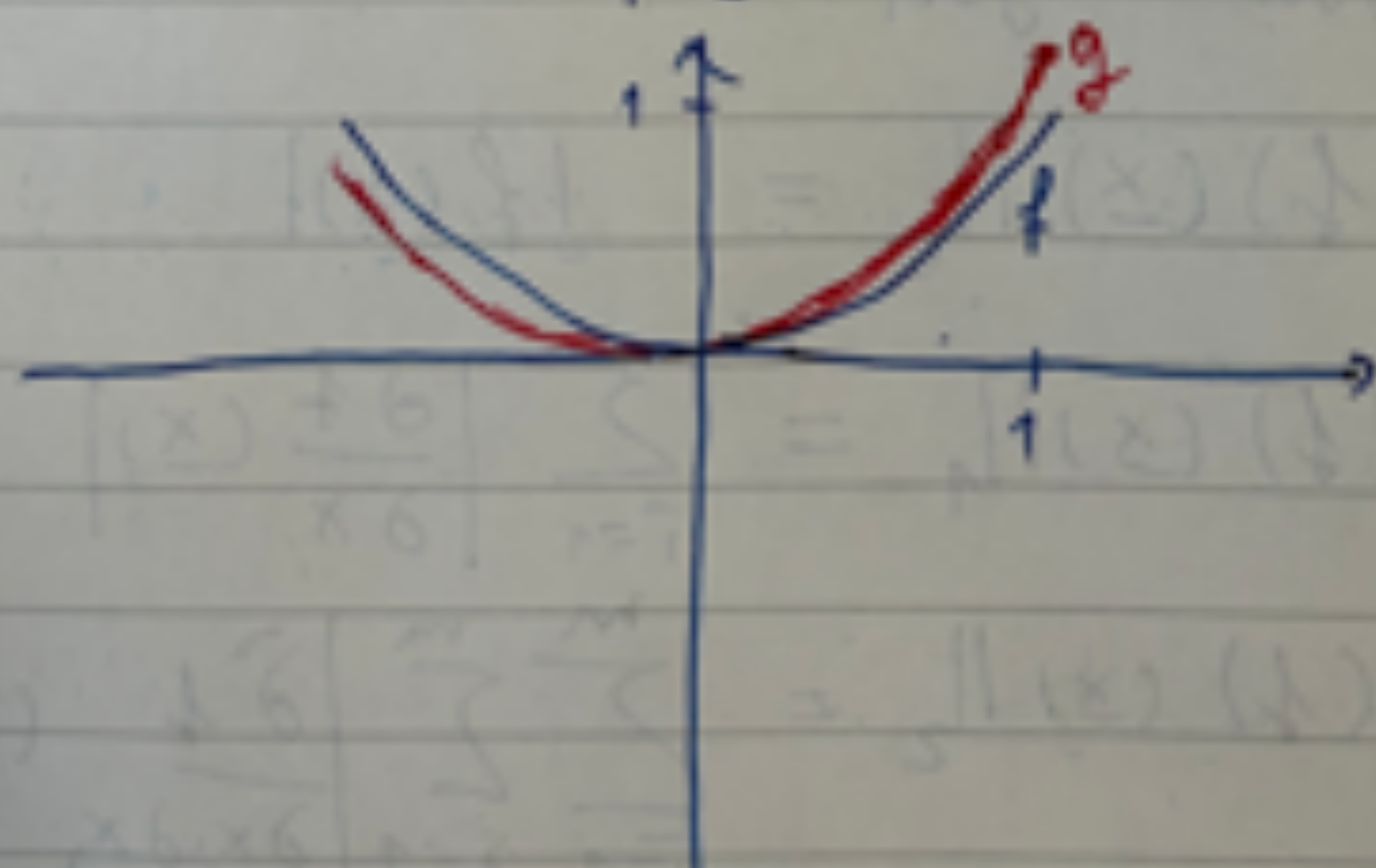
Then $g \in \mathcal{N}^2(f, (\varphi,U), (\psi,V), K, \frac{1}{100})$

and $g \in \mathcal{N}^1(f, \varphi, \psi, K, \frac{1}{200})$

$g \notin \mathcal{N}^2(f, \varphi, \psi, K, \frac{1}{200})$

because $\sup_{[-1,1]} \left| \frac{\partial^2 f}{\partial x^2}(x) - \frac{\partial^2 g}{\partial x^2}(x) \right|$

$= \sup_{[-1,1]} \left| \frac{6}{1000} x \right| = \frac{6}{1000} > \frac{1}{200}$



The weak topology does not control the behaviour at infinity.
$\rightsquigarrow$ We need a finer topology

$\cancel{x_1 \notin [2i + \frac{3}{4}, 2i + \frac{3}{4}]}$

Def 2.10: The strong topology on $C^r(M,N)$, also called Whitney topology, is the coarsest topology containing the following sets:

Take
- $\Phi = (\varphi_i, U_i)_{i \in I}$ a locally finite family of C^r-charts of M
- $\Psi = (\psi_i, V_i)_I$ a set of C^r-charts of N
- $K = (K_i)_{i \in I}$, $K_i \subseteq U_i$ compact
- $f \in C^r(M,N)$, st $f(K_i) \subseteq V_i$ $\forall i \in I$
- $\varepsilon = (\varepsilon_i)_I$, $\varepsilon_i > 0$

$$N^r(f, \Phi, \Psi, K, \varepsilon) := \bigcap_{i \in I} N^r(f, \varphi_i, \psi_i, K_i, \varepsilon_i)$$

Notation 2.11: $C^r(M,N)$ with
- weak topology: $C^r_W(M,N)$
- strong topology: $C^r_S(M,N)$

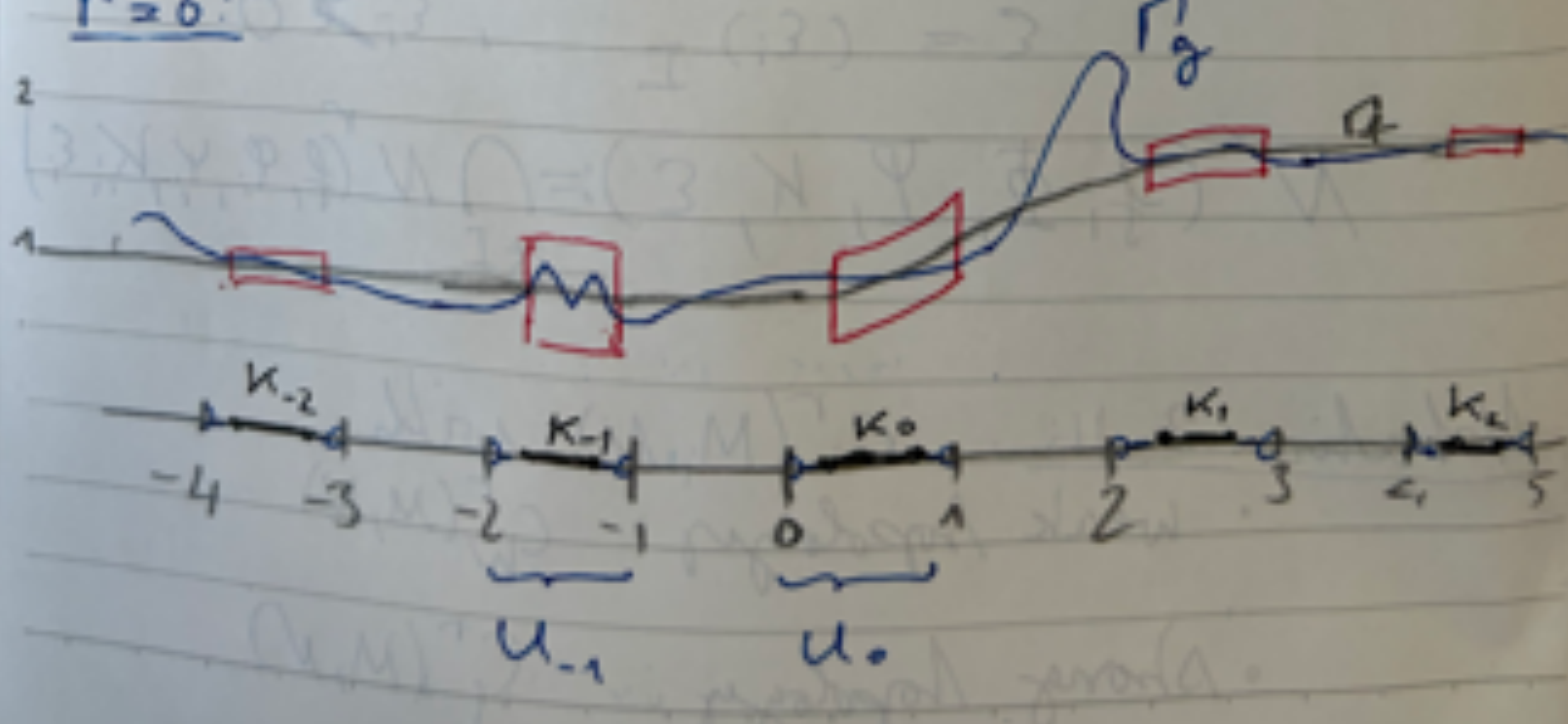
Example 2.12: (i) $M = N = \mathbb{R}$, $\delta > 0$ small

- $I = \mathbb{Z}$, $U_i =]2i, 2i+1[$, $\varphi_i = id$
- $V_i := \mathbb{R}$, $\psi_i = id$
- $K_i := [2i+\delta, 2i+1-\delta]$
- $f \in C^r(M,N)$
- $\varepsilon = (\varepsilon_i)_{\mathbb{Z}}$ $\underbrace{\text{with } \varepsilon_i \downarrow 0 \text{ for } |i| \to \infty}_{\text{just for our example}}$

Take $f(x) = \begin{cases} 1, & x \leq 0 \\ 1 + e^{-\frac{1}{x}}, & x > 0 \end{cases}$

$N^r(f, \Phi, \Psi, K, \varepsilon)$?

r = 0:



$g \in \mathcal{N}^0(f, \Phi, \Psi, K, \varepsilon)$

$g \notin \mathcal{N}^1(f, \Phi, \Psi, K, \varepsilon)$

because $\sup_{x \in K_{-1}} |g'(x) - \underbrace{f'(x)}_{0}| > \varepsilon_{-1}$.

(ii) $C^0_S(\mathbb{R}, \mathbb{R}) = C^0_{\|\cdot\|_\infty}(\mathbb{R}, \mathbb{R})$, i.e. the strong topology on $C^0(\mathbb{R}, \mathbb{R})$ is the $\|\cdot\|_\infty$ " " " .

(Exercise.)

(iii) If M is compact then $C^r_W(M, N) = C^r_S(M, N)$. (Exercise.)

<u>Def 2.12:</u> ($r = \infty$ and $r = \omega$)

(i) The <u>weak</u> topology on $C^\infty(M, N)$ is defined to ~~be~~ be the coarsest such that all inclusions

$$C^\infty(M, N) \hookrightarrow C^r_W(M, N)$$

, for $r<\infty$, are continuous.

$C^\infty_S(M,N)$ is similarly defined.

(ii) The weak / strong topology on $C^\omega(M,N)$ is the subspace topology induced from $C^\infty(M,N)$.

<u>Notation 2.13:</u> $r \geq 1$,

$\mathrm{Imm}^r(M,N)$ = "set of C^r-immersions"
$\mathrm{Subm}^r(M,N)$ = "set of C^r-submersions"
$\mathrm{Emb}^r(M,N)$ = "set of C^r-embeddings"
$\mathrm{Emb}^r_{cl}(M,N)$ = "set of closed C^r-embeddings".
$\mathrm{Diff}^r(M,N)$ = "set of C^r-diffeomorphisms

<u>Theorem 2.14:</u> Let M and N be C^r-mf, $r \geq 1$.
(a) <u>$r\geq 1$</u>: $\mathrm{Imm}^r(M,N)$ and $\mathrm{Emb}^r_{cl}(M,N)$ are open in $C^r_S(M,N)$

(b) $\infty \geq r \geq 1$: $\mathrm{Subm}^r(M,N)$ is open in $C^r_S(M,N)$

(C) Suppose $\partial M = \emptyset = \partial N$. and $\infty \geq r \geq 1$.
Then $\mathrm{Diff}^r(M, N)$ is open in $C^r_S(M, N)$.

Proof: (a) $f \in \mathrm{Imm}^r(M, N)$: Take a locally finite cover $(U_i)_I$ of M with charts (φ_i, U_i). and $K_i \subset U_i$ a compact cover of M.

such that $f(U_i) \subseteq V_{j(i)}$ for some atlas $\Psi = (\psi_j, V_j)_J$ of N.

Then take $\varepsilon_i > 0$ such that for all linear maps $L \in L(\mathbb{R}^m, \mathbb{R}^n)$ with
$$\| D(\psi_j f \varphi_i^{-1})(x) - L \| < \varepsilon_i$$
for some $x \in K_i$
we have that L is injective.
(Why can we find such an ε_i?)

Then $N^r(f, \Phi, \Psi, K, \varepsilon) \subseteq \mathrm{Imm}^r(M, N)$
Thus $\mathrm{Imm}^r(M, N)$ is open.

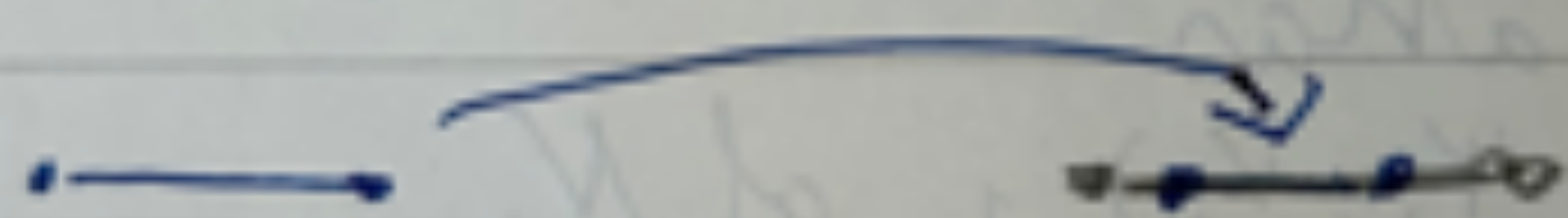
$\mathrm{Emb}^r_{cl}(M, N)$: [II, Cor. 2.1.6.]

(e) Similar to (d). (Exercise.)

(f) $\mathrm{Diff}^r(M,N) = \mathrm{Emb}_{cl}(M,N) \cap \mathrm{Subm}^r(M,N)$

is open in $C^r_S(M,N)$ $\square$

<u>Remark 2.15:</u> 2.14.(c) would be false if we allow boundaries.



$M = [0,1] = N$

$f = id \in \mathrm{Diff}^r([0,1])$

$g_\delta(x) := \delta + x(1-2\delta)$

Then $\lim\limits_{\delta \downarrow 0} g_\delta = f$.

— End of Lecture 10.03.2023