Thm 1.44: ("easy Whitney embedding theorem")

Let M be a compact C^r-mf. with $2 \leq r \leq \infty$. Then there is a C^r-embedding from M into $\mathbb{R}^{2n+1}$.

Remark 1.45: The condition $2 \leq r \leq \infty$ can be improved to $1 \leq r \leq \infty$ by Remark 1.4.

Proof of Thm 1.44: (Sketch of the proof)
See II Thm. 3.5.

Thm 1.42 $\Rightarrow$ $\exists$ $M \hookrightarrow \mathbb{R}^q$ C^r-embedding for some $q \in \mathbb{N}$.
If $q = 2n+1$ ✓
Suppose $q > 2n+1$. We show that M can be embedded into $\mathbb{R}^{q-1}$.
For $v \in S^{q-1} \setminus \mathbb{R}^{q-1} \times \{0\}$, let P_v be the projection onto $\mathbb{R}^{q-1}$ parallel to v.

We need to choose v s.t.

(1) $\forall p, q \in M$, $p \neq q$: $\frac{p-q}{|p-q|_2}$ is not parallel to v.

(2) $\forall (p, w) \in TM \subseteq M \times \mathbb{R}^q$: $\mathbb{R}v \not\ni w$.

Then $P_v|_M : M \longrightarrow \mathbb{R}^{q-1}$ is a C^r-embedding.

We consider the maps σ and ρ:

$\sigma : M \times M - \underbrace{\Delta}_{\{(x,x) \mid x \in M\}} \longrightarrow S^{q-1}$ $\quad \sigma(x,y) := \frac{x-y}{|x-y|_2}$

$\rho : \underset{\{(x,w) \in (M \times \mathbb{R}^q) \cap TM \mid |w|_2 = 1\}}{\overset{\shortparallel}{(TM)_1}} \longrightarrow S^{q-1}$ $\quad \rho(x,w) := w$.

Claim: $((\mathbb{R}^{q-1} \times \{0\}) \cap S^{q-1}) \cup \operatorname{im}\sigma \cup \operatorname{im}\rho \subsetneqq S^{q-1}$.

Proof: Fakt: $\operatorname{im}\sigma$ and $\operatorname{im}\rho$ do not contain any non-empty open subset of S^{q-1}, i.e. we say they have empty interior in S^{q-1}.

Thus $((\mathbb{R}^{q-1} \times \{0\}) \cap S^{q-1}) \cup \operatorname{im}\rho$ has empty interior in S^{q-1} and it is closed in S^{q-1}. This proves the claim $\square$(claim) $\square$

The used lemma is

Lemma 1.46: Let $f \in C^1(M, N)$ and $\dim M < \dim N$. Then $\operatorname{im} f$ has empty interior.

Exercise 1.47: The Lemma in Thm II 13.5 is false.
Find an immersion $c : \mathbb{R} \longrightarrow \mathbb{R}^2$ such that $\overline{\operatorname{im} c} = \mathbb{R}$

We are going to prove Lemma 1.46.

Def 1.48: (a) We say $X \subseteq \mathbb{R}^n$ has measure zero if $\forall \varepsilon > 0$ $\exists (C_i)_{i \in \mathbb{N}}$ cubes : $X \subseteq \bigcup_{i=1}^{\infty} C_i$

and $\sum_{i=1}^{\infty} vol(C_i) \leq \varepsilon$. (write $\lambda_n(X)$

(b) M be a C^1-manifold.
We say that $X \subseteq M$ has measure zero if $\forall \varphi, U \in \mathcal{A}_n$: $\varphi(X \cap U)$ has measure zero.

We need to ensure that Def. 1.4.8 (b) ~~is well-defined~~ can be checked using an atlas.

Remark 1.49.: (1) $X \subseteq \mathbb{R}^n$ has measure zero iff $\forall \underline{x} \in X$ $\exists$ open $U \subseteq \mathbb{R}^n$ wt $\underline{x} \in U$: $U \cap X$ has measure zero. (Why? Exercise.)

(2) Let $X \subseteq \mathbb{R}^n$ be a set of measure zero. Then X does not contain a non-empty open subset. (Exercise.)

Proof of Lemma 1.46 in 2 steps.:

Step 1: Let $U \subseteq \mathbb{R}^n$ be open and $f \in C^1(U, \mathbb{R}^n)$ and $X \subseteq U$ with $\lambda_n(X) = 0$, then $\lambda_n(f(X)) = 0$.

Proof: We can restrict to the case where $\|D(f)\|_2$ is bounded, say by K and U is convex. (Why?) Then we have $|f(\underline{x}) - f(\underline{y})|_2 \leq K \cdot |\underline{x} - \underline{y}|_2$ (Why?) for all $\underline{x}, \underline{y} \in U$.
Then f maps a cube of edge length l into a cube of edge length $l \cdot \sqrt{n} \cdot K$.
If X is covered by cubes C_i with $\sum_{i=1}^{\infty} vol(C_i) \leq \varepsilon$

Then $f(X)$ is covered by cubes C_i' with $\sum_{i=1} \mathrm{vol}(C_i') \leq \varepsilon \cdot K^n$ □

Thus $\lambda_n(f(X)) = 0$.

<u>Step 2:</u> (Proof of the statement of the Lemma)

We can use charts and the second countability to reduce to the case

$$f : \underset{\mathbb{R}^m}{\overset{\cap\!\mid}{U}} \xrightarrow{C^1} \mathbb{R}^n \text{ with } m < n.$$

$U \times \{0\} \subseteq U \times \mathbb{R}^{n-m}$ is a set of measure zero in $\mathbb{R}^n$.

Consider

$$g : U \times \mathbb{R}^{n-m} \longrightarrow \mathbb{R}^n$$
$$g(\cancel{x}, \underline{x}) := f(\underline{x}).$$

By Step 1 the image of g has measure 0.

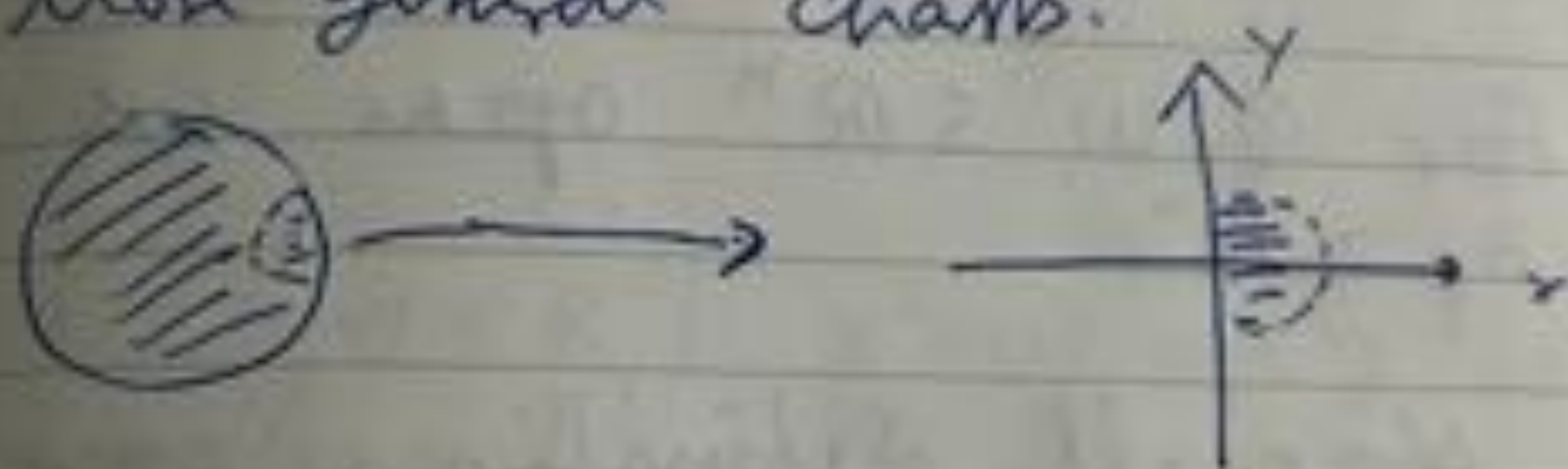
$\Rightarrow \quad \lambda_n(f(U)) = 0$ □

Lecture 5

We need to consider more general manifolds. For example

$$\bar{B}_1(0)_n = \{ \underline{x} \in \mathbb{R}^n \mid |\underline{x}|_2 \leq 1 \} \subseteq \mathbb{R}^n$$

Idea: More general charts:

$\bar{B}_1(0)_2$ 

Def 1.50 (Manifold with boundary)

(a) $H \subseteq \mathbb{R}^n$ is called half space if $\exists\ \lambda \in L(\mathbb{R}^n; \mathbb{R})$:
$H = \{ \underline{x} \in \mathbb{R}^n \mid \lambda(\underline{x}) \geq 0 \}$
We have

$$\partial H = \begin{cases} \{ \underline{x} \mid \lambda(\underline{x}) = 0 \}, & \text{if } \lambda \neq 0 \\ \emptyset, & \text{if } \lambda = 0 \end{cases}$$

(b) We generalize the notion of chart. Let M be a topological space which satisfies Convention 1.10.

A pair (φ, U) with $U \subseteq M$ non-empty open and $\varphi : U \longrightarrow \mathbb{R}^n$ is called a chart if $\exists\ H_\varphi \subseteq \mathbb{R}^n$ a half space such that $\varphi : U \to \varphi(U)$ is a homeomorphism and $\varphi(U)$ ~~$\subseteq H_\varphi$~~ is an open subset of H_φ

For $\varphi(U) \subseteq \mathbb{R}^n$ open we take $H_\varphi := \mathbb{R}^n$.

(c) Now all definitions carry over

(d) Let (φ, U) be a chart. $p \in U$ is called a boundary point of M if $p \in \varphi^{-1}(\partial H_\varphi)$.

(e) $\partial M = \{ p \in M \mid \exists (\varphi, U) \in \mathcal{A}_0$ s.t. p is a boundary point of $(\varphi, U) \}$

where $\mathcal{A}_0$ is a C^0-differential structure of M, i.e. M just seen as a topological manifold.

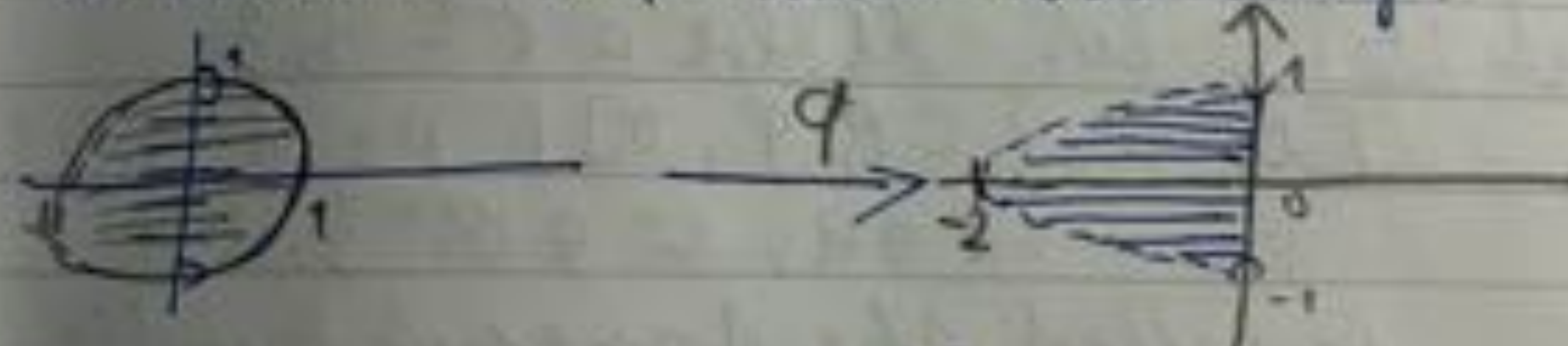
end of Lecture 28.02.2023

Example 1.51: (a) A proper half space is a mf with boundary.

(b) Let $f \in C(\mathbb{R}^{m-1}, \mathbb{R})$ and put $M := \{(\underline{x}, y) \in \mathbb{R}^m \mid y \le f(\underline{x})\}$

$\varphi : M \longrightarrow \mathbb{R}^m$, $\varphi(\underline{x}, y) = (\underline{x}, y - f(\underline{x}))$ is a chart for M.

(c) $D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\}$ is a smooth mf with boundary.



$$\varphi : U \longrightarrow \mathbb{R}^2$$

$$U = \{(x,y) \in D^2 \mid x^2 + y^2 < 1 \text{ if } x \le 0\}$$

Note: U is open in D^2, because $D^2 \setminus U = \{(x,y) \in S^1 \mid x \le 0\}$ is closed.

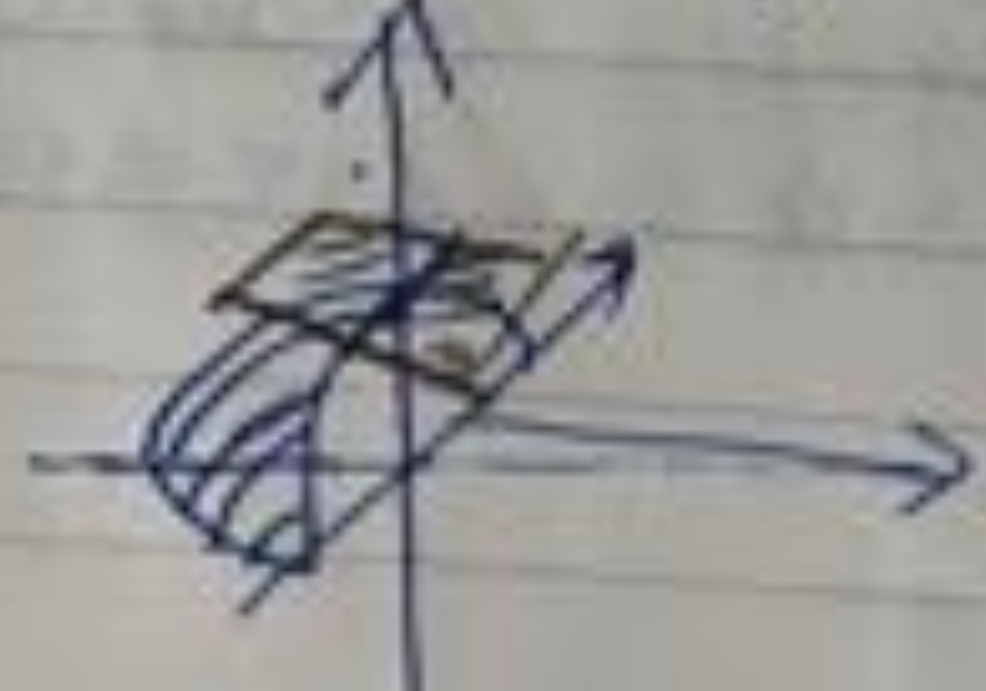
(d) Let M^n be a ~~smooth~~ C^∞ manifold without boundary and $(\varphi, U) \in \mathcal{A}_\infty$. Let B be an open ball in $\mathbb{R}^n$ s.t. $\overline{B} \subseteq U$. Then $M \setminus \varphi^{-1}(B)$ is a mf with boundary and $\partial M = \varphi^{-1}(\partial B)$. (Exercise.)

Def 152: Let M^n be a C^1 mf.

$$TM := \{[p, \varphi, v] \mid p \in M, (\varphi, U) \in \mathcal{A}, p \in U, v \in \mathbb{R}^n\}$$

is called the tangent bundle of M.

For $p \in \partial M$ T_pM is a vector space $\simeq \mathbb{R}^n$.



$p \in \partial M$

$$\underset{\mathbb{R}}{\overset{\wr\!\shortparallel}{T_p \partial M}} \subsetneq T_pM \simeq \mathbb{R}^2$$

Example 1.53: $M = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$

We provide an atlas.

$\varphi_1 : B_1(0) \longrightarrow B_1(0) \subseteq \mathbb{R}^2 \qquad \varphi_1 = id$

$\varphi_2 : M \cap \{(x,y) \mid x > 0\} \longrightarrow \mathbb{R}^2, \quad \varphi_2(x,y) := (x - \sqrt{1-y^2}, y)$

$H_{\varphi_2} = \{(x,y) \mid x = 0\}$

$\varphi_3 : M \cap \{(x,y) \mid y > 0\} \longrightarrow \mathbb{R}^2 \quad \varphi_3(x,y) := (x, y - \sqrt{1-x^2})$

$\left.\begin{matrix}\varphi_4 \\ \varphi_5\end{matrix}\right\}$ Exercise.

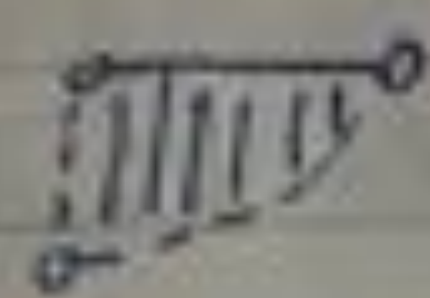
We check the overlaps.

1) $\varphi_1 \circ \varphi_2^{-1} : \varphi_2(U_1 \cap U_2) \longrightarrow \varphi_1(U_1 \cap U_2)$

$\varphi_2(U_1 \cap U_2) = \{(x,y) \in B_1(0) \mid x < 0 \wedge y > 0\}$, $\quad \varphi_1(U_1 \cap U_2) = \{(x,y) \in B_1(0) \mid x > 0 \text{ and } y > 0\}$

$$\varphi_1 \circ \varphi_2^{-1}(x,y) = (x+\sqrt{1-y^2},\, y) \quad C^\infty\text{-map}$$

2) $$\varphi_2 \circ \varphi_3^{-1} : \underset{\{(x,y)\in B_1(0) \mid x>0,\, y\le 0\}}{\underset{\parallel}{\varphi_3(U_2\cap U_3)}} \longrightarrow \underset{\{(x,y)\in B_1(0) \mid x\le 0,\, y>0\}}{\underset{\parallel}{\varphi_2(U_2\cap U_3)}}$$



$$\varphi_2 \circ \varphi_3^{-1}(x,y) = \varphi_2(x,\, y+\sqrt{1-x^2})$$
$$= (x-\sqrt{a(x,y)},\, y+\sqrt{1-x^2})$$

$$a(x,y) := \cancel{x^2 \dots}$$
$$1-(y+\sqrt{1-x^2})^2$$
$$= -y^2+x^2-2y\sqrt{1-x^2}.$$

$$\underset{\sqrt{1-x^2}>-y\ge 0}{>} -y^2+x^2-2y(-y) = x^2+y^2 \underset{x>0}{>} 0$$

$$\varphi_2\circ\varphi_3^{-1} \in C^\infty(\varphi_3(U_2\cap U_3),\, \mathbb{R}^2),$$

because it is a restriction of a C^∞-map on a neighborhood of $\varphi_3(U_1\cap U_3)$.

Tangent bundle of M:

$$TM = \bigsqcup_{p \in \overline{B}_1(0)_2} T_p M$$

$$= \{ [p, \varphi_i, v] \mid p \in M,\ i \in \{1, \dots, 5\},\ \text{s.t. } p \in U_i, \text{ and } v \in \mathbb{R}^2 \}$$

For $p \in \partial M = S^1$ we have $T_p M \cong \mathbb{R}^2$.

From $\varphi_2, \dots, \varphi_5$ we get an atlas for $\partial M = S^1$

It is the one given in Example 1.5 (2).

We need to deliver a definition for $C^\infty(H, \mathbb{R}^m)$

<u>Def 1.54</u>: Let U be an open set in a half space H of $\mathbb{R}^n$ and $0 \le r \le \infty$. Then we define

$$C^r(U, \mathbb{R}^m) := \{ f \in C(U, \mathbb{R}^m) \mid$$

for f all partial derivatives exist ~~an~~ up to order r and are continuous.$\}$

<u>Prop 155:</u> Let H be a half space of $\mathbb{R}^n$ and $U \subseteq H$ be open in H and $0 \leq r \leq \infty$ and $f \in C^r(U, \mathbb{R}^m)$.
Then $\exists\, V \subseteq \mathbb{R}^n$ open s.t. $U \subseteq V$ and $\exists\, g \in C^r(V, \mathbb{R}^m)$: $g|_U = f$.

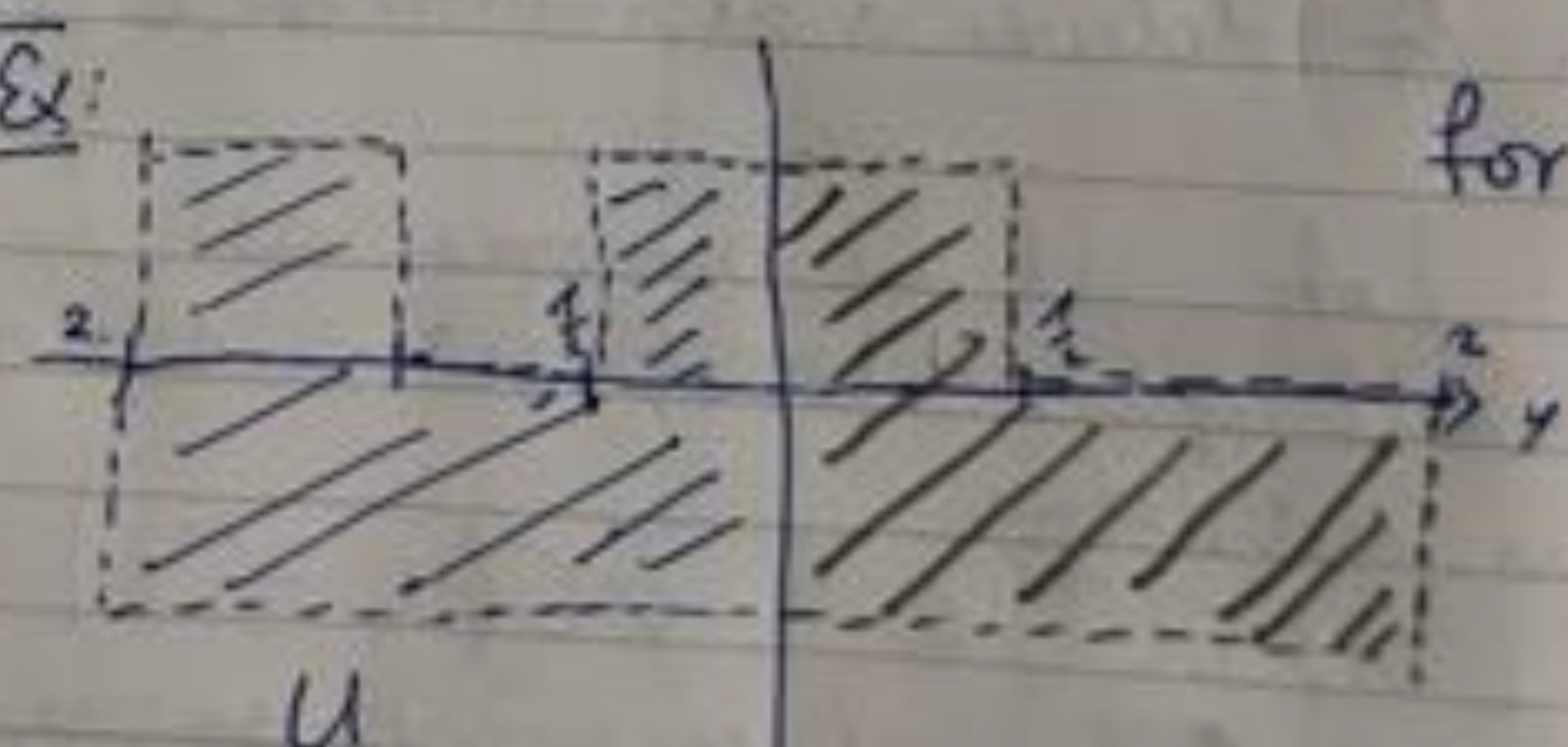
<u>Proof:</u> The case $r = \infty$ is due to Whitney: "Analytic extensions of differentiable functions defined in closed sets"

We just prove the case $0 \leq r < \infty$.
If $H = \mathbb{R}^n$, then ✓.
W.l.o.g. we can assume $H = \mathbb{R}^{n-1} \times \mathbb{R}^{\leq}$:

Put $V := U \cup \{(\underline{x}, y) \in \mathbb{R}^n \mid \cancel{(\underline{x}, -y) \in U}$
$(\underline{x}, -y\frac{1}{i}) \in U$ for $i = 1, \ldots, r+1\}$

Then V is open. (Exercise)

<u>Ex:</u>



for $r \geq 1$.

⌟

Define $g(\underline{x}, y) := \begin{cases} f(\underline{x}, y), & \text{if } (\underline{x}, y) \in U \\ \sum_{i=1}^{r+1} c_i f(\underline{x}, -y \cdot \frac{1}{i}), & \text{if } (\underline{x}, y) \in V \setminus U. \end{cases}$

such that $c_1, \dots, c_{r+1} \in \mathbb{R}$ which satisfy $\sum_{i=1}^{r+1} c_i \left(\frac{-1}{i}\right)^k = 1$, $k = 0, \dots, r$.

Then $g \in C^r(V, \mathbb{R}^m)$ $\square$

Example 1.56: $f \in C^1(]-1, 0], \mathbb{R})$

$f(x) := (1+x)^2$

$U =]-1, 0]$, $V =]-1, 1[$

$$g(x) = \begin{cases} (1+x)^2, & x \in]-1, 0] \\ c_1(1-x)^2 + c_2\left(1 - \frac{x}{2}\right)^2, & x \in]0, 1[\end{cases}$$

$$\begin{aligned} c_1 + c_2 &= 1 \\ -c_1 - \frac{c_2}{2} &= 1 \end{aligned} \iff c_1 = -3 \land c_2 = 4$$

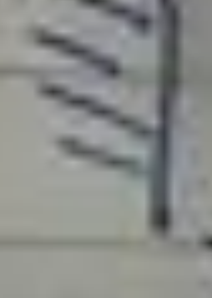
So we get $g(x) = \begin{cases} (1+x)^2, & x \in U \\ \underbrace{-2x^2 + 2x + 1}_{(1+x^2) - 3x^2}, & x \in V \setminus U. \end{cases}$

<u>Question:</u> What should be a submanifold?

By the new problem sheet we have for C^∞-mfs without boundary: $M_1 \subseteq M_2 \subseteq M_3$ with M_2 a submf. of M_3

(*) M_1 is a submf of $M_3 \iff M_1$ is a submf of M_2.

So for more general mf we want:

$N :=$  $\overset{\text{omf}}{\subseteq} \mathbb{R}^2$ and  $\overset{\text{smf}}{\subseteq} \mathbb{R}^2$, $=: H$

and therefore 

N should be a submf of H.

(*) is the key.

Def 1.57: (a) $N \subseteq \mathbb{R}^n$ is called a C^r submanifold of dimension k if for all $p \in N$ there is a C^r-chart of $\mathbb{R}^n$ (φ, U) with $p \in U$ and a half space H of $\mathbb{R}^k$ s.t.:

$$\varphi^{-1}(H) = N \cap U.$$

Ex:

$N := \{ (x, y, 1) \mid (x+2)^2 + (y+2)^2 \leq 1 \}$ is a 1-dim submf. of $\mathbb{R}^3$.

(b) Let M be a C^r mf. and $N \subseteq M^n$. We call N a k-dimensional C^r-submf. of M if $\forall p \in N$ $\exists (\varphi, U) \in \mathcal{A}_r$ with $p \in U$ such that $\varphi(U \cap N)$ is a C^r submf. of $\mathbb{R}^n$. ~~and~~

The notion of immersion, submersion and embedding now carry over to mf with boundary.

For example $f: M \to N$ is called a C^r-embedding if

(c1) $f(M)$ is a C^r-submanifold of N and

(c2) $f: M \to f(M)$ is a C^r-diffeomorphism.

Prop 1.34 is true for general manifolds.

end of Lecture 03.03.2023.

Def. 1.58: Let N be a C^r-submanifold of M^m. (b)

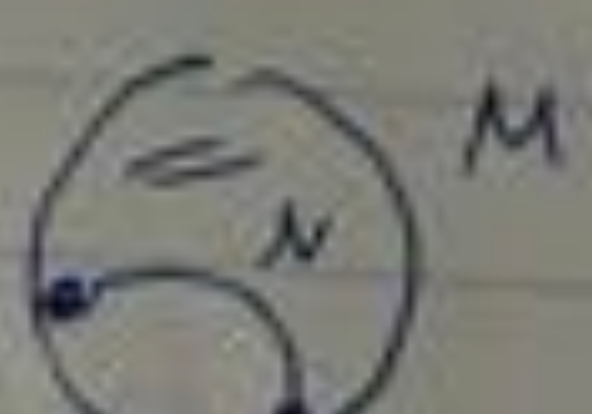
We say N is a <u>neat</u> C^r -"- of M if

(n1) $\partial N = N \cap \partial M$

(n2) $\forall p \in N \; \exists (a, U) \in \mathcal{A}_r$:

$$N \cap U = a^{-1}(\mathbb{R}^n).$$

Example 1.59: (a) We consider C^1 submanifolds of $B_1(0)$



neat

not neat, because $\neg$(n2)