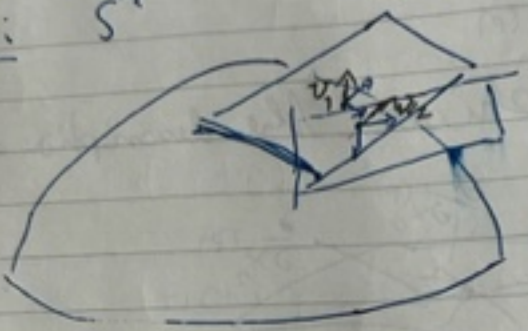
The charts $(T\varphi, TU)$ are called local trivializations of TM.

<u>Ex</u>: S^2



$v_1 = [p_1, \varphi, v]$
$v_2 = [p_2, \varphi, v]$
p_1, p_2 "near".

$[p_1, \varphi, v]$ and $[p_2, \varphi, v_*]$ are "near" but $[p_1, \varphi, v_*]$ and $[p_2, \varphi, v]$ are not. This is the interpretation of

$$TU \simeq \varphi(U) \times \mathbb{R}^n \simeq U \times \mathbb{R}^n$$

topologically.

<u>Def 1.28</u>: Let (M^n) be a smooth manifold

We say that TM can be <u>triviallized</u> if $\exists$ C^{r-1} diffeomorphism

$$\begin{tikzcd} TM \arrow[rr, "\simeq"] \arrow[dr] & & M \times \mathbb{R}^n \arrow[dl] \\ & M \arrow[u, phantom, "\alpha"] & \end{tikzcd}$$

$\mathbb{R}$-linear on fibers.

<u>Ex1.29:</u> (a) $M = S^1$, $r = \omega$

$$TM \simeq S^1 \times \mathbb{R}$$

$$[p, \varphi_\theta, v] \longmapsto (p, v)$$

(exercise)

(b)

$$M := \{ (x, y, z) \in \mathbb{R}^3 \mid$$

$$x = 2\cos\theta + s\cos\tfrac{\theta}{2}\cos\theta$$
$$y = 2\sin\theta + s\cos\tfrac{\theta}{2}\sin\theta$$
$$z = s\sin\tfrac{\theta}{2},$$
$$\theta \in]-\varepsilon, 2\pi + \varepsilon[, \quad s \in]-1, 1[\;\}$$

The ~~...~~ "<u>Möbius strip</u>"

as a submanifold of $\mathbb{R}^3$.

M is a C^∞-manifold s.t. TM is not trivializable.

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<u>Proof:</u> We consider the following two C^∞-maps.

$$\mathbb{R} \xrightarrow{E_1} TM$$
$$\mathbb{R} \xrightarrow{E_2} $$

$$E_2(\theta) = \left(P(\theta, 0), \frac{\partial P}{\partial \theta}(\theta, 0)\right)$$
$$= \left(P(\theta, 0), \begin{pmatrix} -2\sin(\theta) \\ 2\cos(\theta) \\ 0 \end{pmatrix}\right)$$

$$E_1(\theta) = \left(P(\theta, 0), \frac{\partial P}{\partial s}(\theta, 0)\right)$$
$$= \left(P(\theta, 0), \begin{pmatrix} \cos\frac{\theta}{2}\cos\theta \\ \cos\frac{\theta}{2}\sin\theta \\ \sin\frac{\theta}{2} \end{pmatrix}\right)$$

Assume $TM \underset{\Phi}{\overset{C^\infty}{\cong}} M \times \mathbb{R}^2$ a trivialization.

Consider $M \times \mathbb{R}^2 \xrightarrow{pr_2} \mathbb{R}^2$
$$(P, v) \longmapsto v.$$

Then we have for $f_i := \mathrm{pr}_2 \circ \Phi \circ E_i$:
$\mathrm{span}_{\mathbb{R}} \{ f_1(\theta), f_2(\theta) \} = \mathbb{R}^2 \quad \forall \theta \in \mathbb{R}$
and therefore

$$\det \big(f_1(\theta), f_2(\theta) \big) \neq 0 \quad \forall \theta \in \mathbb{R}$$

(Exercise.)
~~But~~ Now $\det (f_1(0), f_2(0))$

$$= -\det(-f_1(0), f_2(0))$$
$$= -\det(f_1(2\pi), f_2(2\pi))$$

and thus the image of $\theta \in \mathbb{R} \xmapsto{\det(f_1, f_2)} \det(f_1(\theta), f_2(\theta))$ must be disconnected.
A contradiction, because $\mathbb{R}$ is connected and $\det(f_1, f_2)$ is continuous. $\square$

<u>Def</u> <u>1.30</u> Let $(M^n, \mathcal{A}_r)$ be a smooth manifold, i.e. $r \geq 1$.
A map $X \in C^{r-1}(M, TM)$ such that $\forall p \in M \; X(p) \in T_pM$ is called a <u>C^{r-1}-vector field on M</u>

Prop 1.31 Let $(M^n, \mathcal{A}_r)$ be a smooth manifold. Then are equivalent

1° TM can be trivialized

2° $\exists\, X_1, \dots, X_n$ C^{r-1} vector fields on M such that for all $p \in M$ we have

$$\operatorname{span}_{\mathbb{R}}\{X_1(p), \dots, X_n(p)\} = T_pM$$

Proof: Exercise. □

Lecture 4.

We now study certain important maps between manifolds.

Def 1.32: Let M and N be C^r-manifolds $(r \geq 1)$, $f \in C^1(M, N)$ and $p \in M$.

(a) The map f is called immersive (submersive) at p if

$$T_pf : T_pM \longrightarrow T_{f(p)}N$$

is injective (surjective).

(b) The map f is called an <u>immersion</u> (a submersion) if f is immersive (submersive) at every point of M.

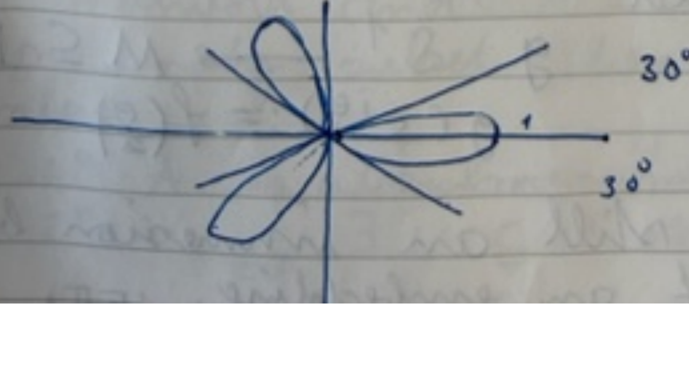
(c) Suppose f is a C^r-map. It is called a C^r-<u>embedding</u> if

(c1) $f(M)$ is a C^r-submanifold of N and

(c2) $f: M \longrightarrow f(M)$ is a C^r-diffeomorphism.

<u>Example 1.33</u>: (a) Consider the curve given by $r = \cos(3\theta)$ in extended polar coordinates (i.e. we allow negative radius with $(x(r,\theta), y(r,\theta)) = (x(-r, \theta+\pi), y(-r, \theta+\pi))$)

$$M := \{ (\cos\theta \cdot \cos(3\theta), \sin\theta \cdot \cos(3\theta)) \mid \theta \in \mathbb{R} \} = \{ P(\theta) \mid \theta \in \mathbb{R} \}$$



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The map $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ defined via $f(\Theta) := P(\Theta)$. is an immersion, but not an embedding, because not injective.

Proof:

$$f'(\theta) = \begin{pmatrix} -\sin\theta\cos(3\theta) - 3\cos\theta\sin 3\theta \\ \cos\theta\cos(3\theta) - 3\sin\theta\sin 3\theta \end{pmatrix}$$

Assume $\exists\, \theta \in \mathbb{R}: \; f'(\theta) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Then
$$\begin{aligned} &\mathrm{I} \quad -\sin\theta\cos(3\theta) = 3\cos\theta\sin(3\theta) \\ &\mathrm{II} \quad \cos\theta\cos(3\theta) = 3\sin\theta\sin(3\theta) \end{aligned}$$

$\cos\theta\,\mathrm{I} + \sin\theta\,\mathrm{II}: \quad 0 = 3\sin(3\theta)$

$-\sin\theta\,\mathrm{I} + \cos\theta\,\mathrm{II}: \quad \cos(3\theta) = 0$

$\Rightarrow \quad \sin(3\theta) = \cos(3\theta) = 0 \quad ↯ \quad \square$

(b) In (a) we could have considered

$$g : S^1 \longrightarrow M \subseteq \mathbb{R}^2$$
$$g(e^{i\theta}) := f\left(\tfrac{\theta}{2}\right).$$

g is still an immersion but not an embedding.

$\left(g(e^{i\frac{\pi}{3}}) = g(e^{i\pi}) = g(e^{i5\frac{\pi}{3}}) = (0,0) \right)$

But injective immersions are not so far away from being embeddings.

Prop. 1.34: Let $f \in C^r(M^m, N^n)$ $(r \geq 1)$ be injective. Then are equivalent

1° f is a C^r-embedding

2° (2°.a) f is an immersion and
(2°.b) $f: M \longrightarrow f(M)$ is a homeomorphism.

Remark 1.35: If M is compact then (2°.b) is automatically satisfied, because $f(M)$ is Hausdorff and $f: M \longrightarrow f(M)$ is bijective and continuous.

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Lemma 1.36: Let (M, α_r) be a C^r-manifold and $A \subseteq M$ a non-empty subset. Then are equivalent:

1° A is a C^r-submanifold of M

2° $\forall_{p \in A}$ $\exists$ $U \subseteq M$ open with $p \in U$: $A \cap U$ is a C^r-submanifold of U.

<u>Proof:</u> Exercise □

<u>Proof of Proposition 1.34.:</u>

$1° \Rightarrow 2°$ Exercise.

$2° \Rightarrow 1°$ By Lemma 1.36., to
(c1) prove that $f(M)$ is a C^r-submanifold
we can restrict to the case

$$f : \underset{\underset{\mathbb{R}^m}{\cap|\text{open}}}{V} \longrightarrow \mathbb{R}^n, \; f(\underline{0}) = \underline{0}.$$

$$D(f)(\underline{0}) = \left(\frac{\partial f_j}{\partial x_i} \right)_{\substack{(j,i) \\ 1 \le j \le n \\ 1 \le i \le m}} (\underline{0})$$

has full rank m by (2°. a).
W.l.o.g. $\left(\frac{\partial f_j}{\partial x_i}(\underline{0}) \right)_{\substack{(j,i) \\ 1 \le j \le m \\ 1 \le i \le m}}$

has non-vanishing determinant.

$g := f_{\underline{x}} : V \longrightarrow \mathbb{R}^m$

$$\left(f_{\underline{x}} = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \quad f_{\underline{y}} = \begin{pmatrix} f_{m+1} \\ \vdots \\ f_n \end{pmatrix} \right)$$

Inverse function theorem $\Rightarrow \exists_{W \subseteq V}$ and $Z \subseteq \mathbb{R}^m$ both open with $\underline{0} \in W \cap Z$ such that

$g|_W : W \longrightarrow Z$ is a C^r-diffeomorphism.

We now find a submanifold chart for $f(V)$ around $\underline{0} \in \mathbb{R}^n$.

$$U := (Z \times \mathbb{R}^{n-m}) \setminus \overline{f(V \setminus W)}$$

closure in $\mathbb{R}^n$.

Claim: $U \cap f(V \setminus W) = \emptyset$ and $U \supseteq f(W)$.

Proof: $w \in W$. Assume $f(w) \notin U$

$\Rightarrow \exists\, v_n \in V \setminus W : f(v_n) \longrightarrow f(w)$

$\underset{f \text{ homeomorphism}}{\Rightarrow}$ $v_n \longrightarrow w$ in V

$\underset{V \setminus W \text{ is closed in } V}{\Rightarrow}$ $w \in V \setminus W$ $\lightning$ $\square$

$\varphi : U \longrightarrow \mathbb{R}^n$

$\varphi(\underline{z}, \underline{y}) := (\underline{z}, \underline{y} - f_{\underline{y}}(g^{-1}(\underline{z})))$

$$D(\varphi)(\underline{z}, \underline{y}) = \begin{pmatrix} I_m & 0 \\ * & I_{n-m} \end{pmatrix}$$

has full rank.

$\Rightarrow \varphi : U \longrightarrow \varphi(U)$ is a C^r-diffeomorphism.

$$\varphi^{-1}(\mathbb{R}^m \times \{\underline{0}\}) = \{(\underline{z}, \underline{y}) \in U \mid \exists \underline{w} \in W : f(\underline{w}) = (\underline{z}, \underline{y})\}$$

$$= f(W) \cap U \underset{\text{Claim}}{=} f(V) \cap U. \quad \square \; c(c1)$$

c(2) is implied by the Inverse function theorem. $\square$

Example 1.37.: The map

$f : S^1 \longrightarrow S^1 \times S^1$

$e^{i\theta} \longmapsto (e^{i\theta}, e^{2i\theta})$

is an embedding, because $T_{e^{i\theta_0}} f$ has in θ-coordinates

the form $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbb{R}^{1\times 2}$ and

The C^r-map f is injective.
We apply Remark 1.35.

<u>Question 138</u>: Suppose $f \in C^r(M,N)$ is an injective immersion and M is not compact. Is f an C^r-embedding?
(Exercise.)

<u>Def 1.39</u>: Let $f \in C^1(M,N)$ and $P \in M$ and $Q \in N$.

(a) We call P a <u>regular point</u> for f if f is submersive at P.
Otherwise we call P a <u>critical point</u> of f.

(b) The point Q is called a <u>regular value</u> if all elements of $f^{-1}(Q)$ are regular points.

(c) The point Q is called a <u>critical value</u> if $f^{-1}(Q)$ contains a critical point.

(d) If Q is a regular value of f and $Q \in f(M)$ then we call $f^{-1}(Q)$ a <u>regular level surface</u>.

Remark 1.40: By 1.39(b) the elements of $N \setminus f(M)$ are also regular values, even if they are not values of f.

Theorem 1.41: (Regular value theorem) Let $f \in C^r(M,N)$, $r \geq 1$, and $Q \in f(M)$ be a regular value of f. Then $f^{-1}(Q)$ is a C^r-submanifold of M. of dimension $\dim M - \dim N$.

Proof: Exercise. □

Lecture 5

We now prove that compact smooth manifolds can be viewed as submanifolds of Euclidean spaces.

Theorem 1.42: Let M be a C^r-manifold for some $1 \leq r \leq \infty$. Then $\exists q \in \mathbb{N} \, \exists f: M \longrightarrow \mathbb{R}^q$: f is a C^r-embedding.

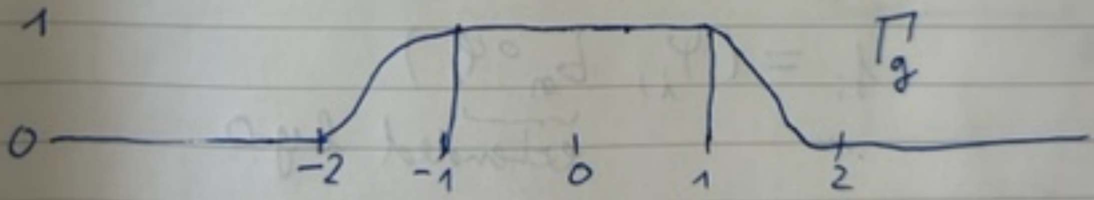
For the proof we need the following objects:

- a bump function $g : \mathbb{R} \to [0,1]$

$$g(t) := \begin{cases} 1 & , \ t \in [-1,1] \\ 0 & , \ t \in \,]-\infty,-2] \cup [2,\infty[\\ f(2-|t|) & , \ t \in [-2,-1] \cup [1,2] \end{cases}$$

with

$$f(t) := \begin{cases} e^{1 - \frac{1}{1 - t^2 \exp(1 - \frac{1}{t})}} & , \ t \in \,]0,1[\\ 1 & , \ t \in \,]-\infty, 0] \\ 0 & , \ t \in [1, \infty[\end{cases}$$



This gives bump functions $b_n : \mathbb{R}^n \to [0,1]$
$b_n(\underline{x}) := g(|\underline{x}|_2)$, $n \in \mathbb{N}$.

We also need in $\mathbb{R}^n$
$B_s(x_0) = \{ x \in \mathbb{R}^n \mid |\underline{x} - x_0|_2 < s \}$
ball around x_0 with radius s.

Proof of Theorem 1.42:

M is compact $\Rightarrow \exists m \in \mathbb{N}\ \exists$ C^r-charts $(\varphi_1, U_1), \dots, (\varphi_m, U_m)$ such that:

- $\varphi_i(U_i) \supseteq B_3(\underline{0})$
- $\bigcup_{i=1}^{m} \varphi_i^{-1}(B_1(\underline{0})) = M.$

Define $\psi_i(p) = \begin{cases} \varphi_i(p)\, b_n(\varphi_i(p)), & p \in U_i \\ 0 & , p \notin U_i \end{cases}$

and $f_i : M \longrightarrow \mathbb{R}^{n+1}$ via

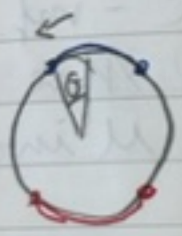
$$f_i = (\psi_i,\ \underbrace{b_n \circ \varphi_i}_{\text{extended by } 0})$$

and $f : M \longrightarrow \underbrace{\mathbb{R}^{n+1} \times \dots \times \mathbb{R}^{n+1}}_{m}$

$$f := (f_1, \dots, f_m) \qquad \square$$

<u>Example 1.43:</u> How does the recipe of the proof of Thm. 1.42 embeds S^1 into $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$.?

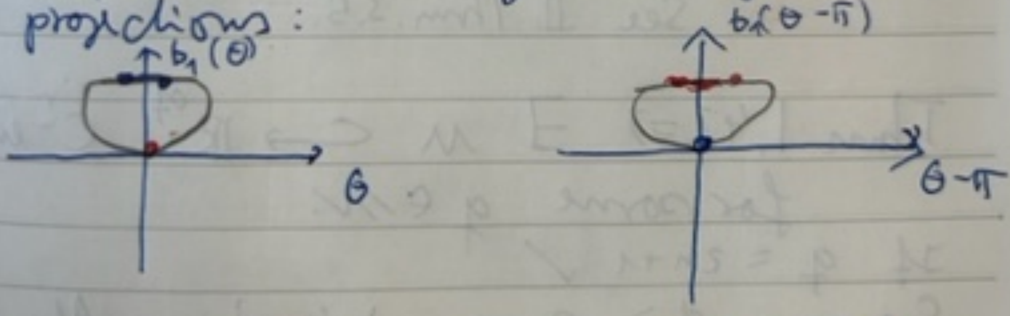
S^1



φ : angle chart on the complement of the blue set

ψ : angle chart... red set.

$S^1 \longrightarrow \mathbb{R}^2 \times \mathbb{R}^2$ is given by the two projections:



In fact we know that $S^1 \subseteq \mathbb{R}^2$, so the q in Thm. 1.42 is not optimal.

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