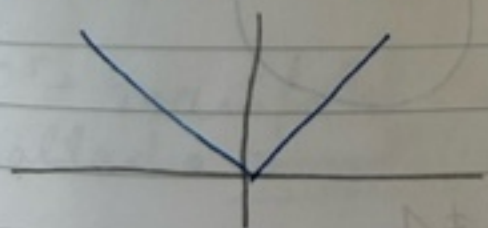
Lecture 1 7.2.2023

Chapter 1 Manifolds and submanifolds.

Def 1.1: Let X be a topological space. $n \in \mathbb{N}_0$
X is called an n-dimensional if for every $x \in X$
$\exists$ $U \subseteq X$ open with $x \in U$ and
a map $\varphi : U \longrightarrow \mathbb{R}^n$
such that φ maps U homeomorphically to $\varphi(U)$ and $\varphi(U)$ is open in $\mathbb{R}^n$.

We call (φ, U) a chart with domain U.

Example 1.2: a) $X = \{ (x, |x|) \in \mathbb{R}^2 \mid x \in \mathbb{R} \}$



Here one chart is sufficient to cover X

$\varphi : X \longrightarrow \mathbb{R}$ $\varphi(x, |x|) = x$

The inverse $z \longmapsto (z, |z|)$ is continuous too. Thus (φ, X) is a chart and X is a ~~topological~~ 1-dimensional manifold

b) $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

Take $(x_0, y_0) \in S^1$

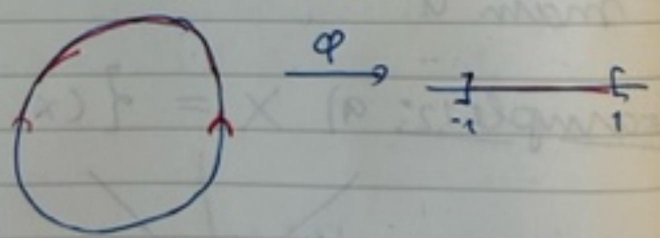
Case $|x_0| \neq 1$

$$\varphi : \{(x,y) \in S^1 \mid y y_0 > 0\} \longrightarrow \mathbb{R}$$
$$\varphi(x,y) := x$$

is a chart because $\mathrm{im}(\varphi) =]-1, 1[$ and φ and its inverse

$$x \longmapsto (x, \mathrm{sgn}(y_0)\sqrt{1-x^2})$$

are continuous.



Case $|y_0| \neq 1$

$$\psi : \{(x,y) \in S^1 \mid x x_0 > 0\} \longrightarrow \mathbb{R}$$
$$\psi(x,y) := y$$

is a chart.

Def. 1.3: Let X be an n-dim manifold. Two charts ~~with~~ $(\varphi, U), (\psi, V)$ are said to have a C^r overlap if either $U \cap V = \emptyset$ or $V \cap U \neq \emptyset$ and $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ is a C^r-diffeomorphism. $(0 \leq r \leq \infty, \omega)$

A family of charts $\{\varphi_i, U_i\}_{i \in I}$ of X is called an atlas if $X = \bigcup_{i \in I} U_i$

An atlas of X is called C^r if every pair of charts has a C^r overlap.

A maximal C^r-atlas $\mathcal{A}$ on X is called a C^r-differential structure, and $(X, \mathcal{A})$ is called a C^r-manifold.
A C^r-manifold with $r \geq 1$ is called a smooth manifold.

Remark 1.4: (a) We may see later (if time allows) that every smooth manifold $(M, \mathcal{A})$ has a unique C^∞-differential structure, i.e. $\exists! \alpha_\infty$ on M: $\alpha_\infty \cap \alpha \neq \emptyset$

(b) There C^0-manifolds M, s.t. ~~there is no~~ doesn't exist any C^1-differential structure on M. (non-trivial statement)

Example 1.5.

(1) Let $f: \underset{\substack{\cap \text{open} \\ \mathbb{R}^n}}{D} \longrightarrow \mathbb{R}^m$ be a continuous map

$\Gamma_f = \{(x, f(x)) \mid x \in D\} \subseteq D \times \mathbb{R}^m$
"graph of f"

The chart $\varphi_1: \Gamma_f \xrightarrow{(x, f(x)) \mapsto x} D \subseteq \mathbb{R}^n$

gives an analytic differential structure, i.e. $\exists$ an C^ω-diff str. α_ω on Γ_f, s.t. $(\varphi_1, \Gamma_f) \in \alpha_\omega$.

Question: If f is not C^1. How can we see this in terms of ~~f~~ manifolds using Γ_f?

(2) $S^n := \{ \underline{x} \in \mathbb{R}^{n+1} \mid |\underline{x}|_2 = 1 \}$

The following ~~set of~~ charts form a C^ω- atlas on S^n.

- For $s \in \{\pm 1\}$ and $i \in \{1, \dots, n+1\}$ we define

$$\varphi_{i,s} : U_{i,s} \longrightarrow \mathbb{R}^n$$

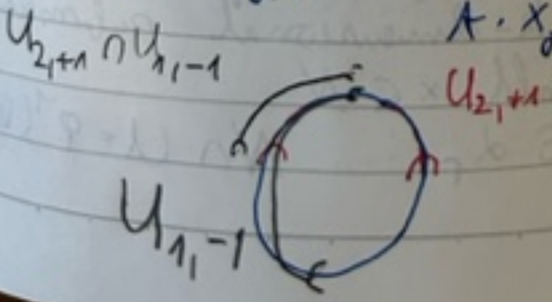
via $\varphi_{i,s}(\underline{x}) = (x_1, \dots, \hat{x}_i, \dots, x_{n+1})$

and $U_{i,s} := \{ \underline{x} \in S^n \mid s \cdot x_i > 0 \}$

- Check overlaps: Take (i,s) and ~~cases:~~ (j,t) s.t. $i \neq j$.

$$(\varphi_{i,s} \circ \varphi_{j,t}^{-1})(\underline{z}) = (z_1, \dots, \overset{(i)}{\hat{z}_i}, \dots, \sqrt{1 - |\underline{z}|_2^2} \cdot t, \dots, z_n)$$

$$U_{i,s} \cap U_{j,t} = \{ \underline{x} \in S^n \mid s \cdot x_i > 0 \text{ and } t \cdot x_j > 0 \}$$



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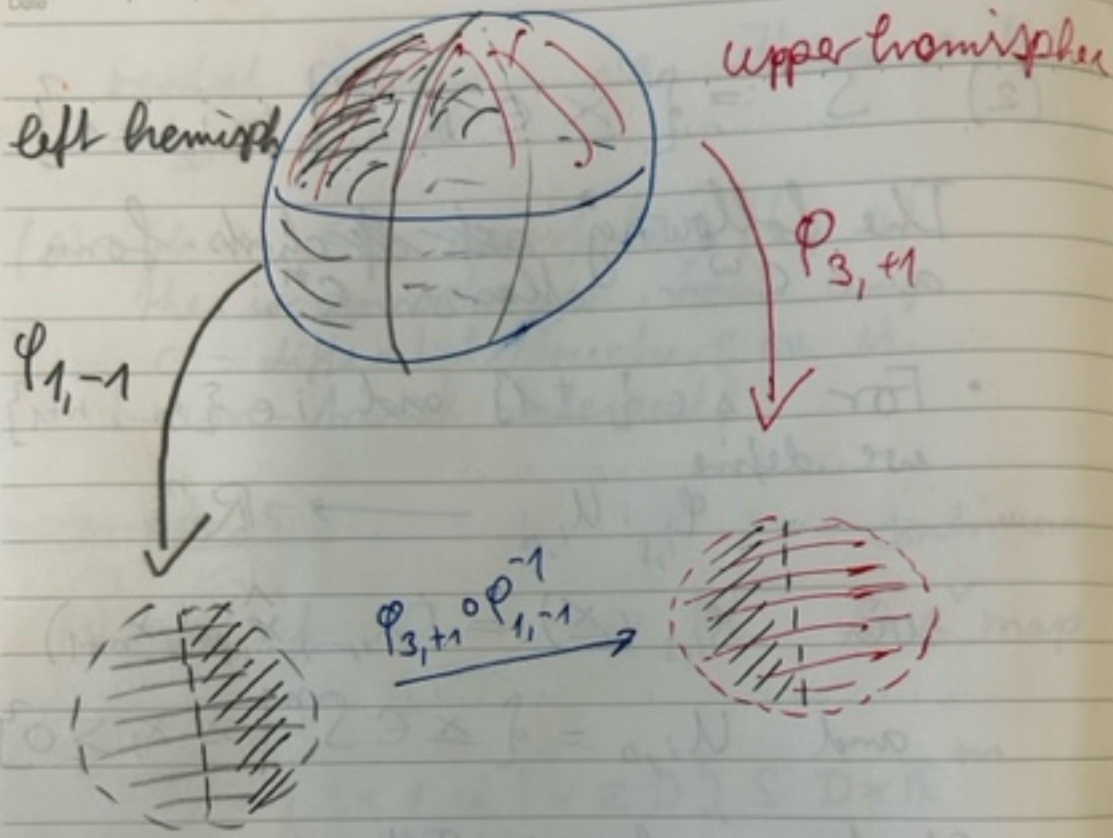

<u>Def 1.6:</u> (submanifold)

Let M^n be a manifold with a C^r-differential structure $\mathcal{A}_r$. ($r \geq 0$)
Let N be a non-empty subset of M.
N is called a k-dimensional submanifold of M if for all $x \in N$

$$(*) \left\{ \exists\, (\varphi, \underset{\substack{\Psi \\ x}}{U}) \in \mathcal{A}_r : \quad N \cap U = \varphi^{-1}(\mathbb{R}^k \times 0) \right.$$

A chart of M satisfying $(*)$ is called a submanifold chart for N.

Remark 1.7: If N^k is a submanifold of $(M^n, \mathcal{A}_r)$ then the set
$$\{(\varphi|_{U\cap N}, U\cap N) \mid \underbrace{(\varphi, U) \in \mathcal{A}_r}_{\text{submf. chart}}, U\cap N \neq \emptyset\}$$
is a C^r- atlas of N.
Therefore $(N, \mathcal{A}_r|_N)$ is a manifold,

$\mathcal{A}_r|_N$ being the C^r- differential structure containing the above atlas.

Examples 1.8: (a) $N = \{(x,y,z) \in \mathbb{R}^3 \mid z = x^2+y^2\}$ is a C^ω-submanifold of $\mathbb{R}^3$, because $(\varphi, \mathbb{R}^3)$ def. via $\varphi(x,y,z) := (x, y, z-x^2-y^2)$ is a submanifold chart for N;

- $\varphi: \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism
- $(\varphi, \mathbb{R}^3)$ has a C^ω- overlap with $(id, \mathbb{R}^3)$
- $\varphi^{-1}(\mathbb{R}^2 \times \{0\}) = N$.

(b) $N = \{(x, |x|) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ is not a C^1- submanifold of $\mathbb{R}^2$. (Exercise.)

(c) S^n is a C^ω-submanifold of $\mathbb{R}^{n+1}$. (Same technique as in (a).)

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<u>Remark 1.9</u>: (a) If M is a smooth manifold and let (φ, W) and (ψ, V) be two charts with a C^1-overlap then the dimensions of the target spaces $\mathbb{R}^{n_\varphi}$, $\mathbb{R}^{n_\psi}$ are the same. (Use Jacobi matrices)

(b) We have more:
Let M be an n-dimensional manifold (maybe just topological) and

$$\psi : V \hookrightarrow \mathbb{R}^m$$

be an open embedding of an non-empty open set of M. Then $m = n$.

(because a non-empty subset of $\mathbb{R}^m$ is homeomorphic to an open set of $\mathbb{R}^m$ if and only if it is open in $\mathbb{R}^m$.)

Therefore we call n the dimension of M.

Convention 1.10: We only conside manifolds which are Hausdorff, second countable and paracompact. ~~[illegible]~~

Reminder 1.11: A topological T^2-space (X,τ) is called paracompact if for every open cover $\mathcal{C} \subseteq \tau$ of X there is a refinement $\mathcal{S}$ of $\mathcal{C}$ (i.e. $\mathcal{S}$ is an open cover s.t. $\forall u \in \mathcal{S} \exists v \in \mathcal{C}: u \subseteq v$) which is locally finite (i.e. $\forall x \in X$ $\exists K \subseteq X$, a ~~[illegible]~~ neighbourhood of x, s.t. only finitely many elements of $\mathcal{S}$ intersect K)

Theorem 1.12: (Smirnov's metrization theorem)
Let X be a T^2-space. Then are equivalent

1° X is metrizable

2° X is paracompact and locally metrizable, i.e. $\forall x \in X \exists U \in \tau: x \in U$ and U is metrizable.

By Theorem 1.12 and Convention 1.10 every manifold will be metrizable.

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<u>Convention 1.13:</u> If we say "differentiable structure" then we mean C^r – " – – " – with $r > 0$.

Now we are able to define what it means for maps to be differentiable.

<u>Def 1.14:</u> Let (M, α_r) and (N, β_r) be smooth manifolds and $f \in Map(M, N)$.

(a) We call f differentiable at $x_0 \in M$ if $\exists$ $(\varphi, U) \in \alpha_r$ with $x_0 \in U$ and $(\psi, V) \in \beta_r$ with $f(U) \subseteq V$: $\psi \circ f \circ \varphi^{-1} : \varphi(U) \longrightarrow \psi(V)$ is differentiable at $\varphi(x_0)$.

(b) We call f r-times $(r < \omega)$ differentiable at x_0 if ... – " – ... :
$\psi \circ f \circ \varphi^{-1} \in C^{r-1}(\varphi(U), \psi(V))$
and $\psi \circ f \circ \varphi^{-1}$ is differentiable at $\varphi(x_0)$

(c) We call f differentiable of class C^r (or C^r-map) if
$\forall (\varphi, U) \in \alpha_r$ $\forall (\psi, V) \in \beta_r$ with $f(U) \subseteq V$: $\psi \circ f \circ \varphi^{-1} \in C^r(\varphi(U), \psi(V))$

For $r = \omega$ we call f analytic.

(d) A continuous map f is called C^0-map (or of class C^0

<u>Examples 1.15:</u> (a) A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is of class C^r in the sense of Def 1.14(c) iff $f \in C^r(\mathbb{R}^n, \mathbb{R}^m)$. (Just take the charts $(\mathrm{id}_{\mathbb{R}^n}, \mathbb{R}^n)$ and $(\mathrm{id}_{\mathbb{R}^m}, \mathbb{R}^m)$ to verify)

(b) Every C^r-map $S^n \longrightarrow M$ is a restriction of a C^r-map $\mathbb{R}^{n+1} \setminus \{0\} \to M$

$\ulcorner$ Given $f : S^n \longrightarrow M$ of class C^r

define $F : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow M$ via
$F(\underline{x}) := f(\underline{x} \cdot \frac{1}{|\underline{x}|_2})$. $\lrcorner$

<u>Question:</u> Can we extend every C^r-map $S^n \longrightarrow M$ to a C^r-map $\mathbb{R}^{n+1} \longrightarrow M$?
At least if $M = \mathbb{R}^k$? (Exercise!)
(Consider the case $r \neq \omega$ and $r = \omega$.)

(c) Consider $M := \{(x, |x|) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$

The map $\mathbb{R} \longrightarrow M$ $\quad x \mapsto (x, |x|)$ is a C^ω-map between the C^ω-manifolds $\mathbb{R}$ and M.

<u>Notation 1.16:</u> Let M, N be C^r-manifolds. We denote by $C^r(M,N)$ the set of C^r-maps from M to N.

Lecture 2

<u>Def. 1.17:</u> (tangent vector)

(a) Let M^n be a C^1 manifold.
We define an equivalence relation on "$M \times \mathcal{A}_1$" $\times \mathbb{R}^n$:

$$(x, \varphi, v) \sim (y, \psi, w) \iff x = y \text{ and } D(\varphi\psi^{-1})(\psi(x)) \cdot w = v.$$

The equivalence class $[x, \varphi, v]$ is called a <u>tangent vector</u> of M at x.

The whole factor set

$$\text{"}M \times \mathcal{A}_1\text{"} \times \mathbb{R}^n \big/ \sim$$

is called the <u>tangent bundle</u> of M.

(b) A map $f \in C^1(M, N)$ defines a ~~ta~~ map $Tf : TM \longrightarrow TN$ via

$$Tf([x, \varphi, v]) := [f(x), \psi, D(\psi f \varphi^{-1})(\ldots$$

v

(for (φ, U) and (ψ, V) s.t. $f(U) \subseteq V$.),
called the <u>derivative of f</u>.

(c) $T_x M := \{ [x, \varphi, v] \mid \varphi \in \mathcal{A}_1, v \in \mathbb{R}^n \}$
is called the tangent space of M at x.
($T_x M \cong \mathbb{R}^n$ as vector space : (φ_0, U_0)
a chart around x.
$[x, \psi, w] \longmapsto D(\varphi\psi^{-1})(\psi(x))w$)

(d) For $f \in C^1(M, N)$ the restriction of Tf
to $T_x M$: $T_x f : T_x M \longrightarrow T_{f(x)} N$

~~if~~ is called the derivative of f at x.

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<u>Examples 1.18</u>:

(a) The inclusion $S^n \xrightarrow{\iota} \mathbb{R}^{n+1}$
is analytic. $(n \geq 1)$, where
we consider on S^n the atlas of
Ex. 1.5.(2).
(We need to specify as there are
differentiable structures on S^7
which are not compatible with
Ex. 1.5.(2).)
$1 \leq i \leq n+1$, $\varepsilon \in \{\pm 1\}$.