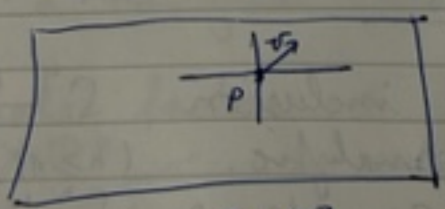
$$id_{\mathbb{R}^{n+1}} \circ \varphi_{i,\varepsilon}^{-1}(\underline{z}) = \Big(z_1, \dots, \overset{(i)}{\varepsilon\sqrt{1-|\underline{z}|_2^2}}, \dots, z_n\Big)$$

is C^∞ because $|\underline{z}|_2 \neq 0$ on $\mathrm{im}(\varphi_{i,\varepsilon})$.

(b) How does the definition of tangent space at a point coincides with the intuition?

Example: $S^1 \hookrightarrow \mathbb{R}^2$

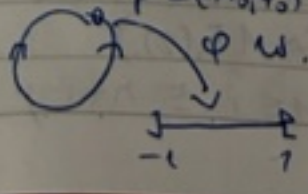
For $\mathbb{R}^2$: At every $p \in \mathbb{R}^2$ is attached a tangent space $\simeq \mathbb{R}^2$



tangent bundle $\simeq \coprod_{p \in \mathbb{R}^2} (p, p+\mathbb{R}^2)$

For S^1 $\quad p=(x_0,y_0) \quad x_0^2+y_0^2=1$

φ w.l.o.g. $y_0>0$.



Formal definition: Take (φ, U) around p

$(\varphi_{2,+1}, U_{2,+1})$

$\varphi(p) \in]-1,1[\subseteq \mathbb{R}$ has attached a tangent space $\varphi(p)+\mathbb{R}$.

Take $v \in \mathbb{R}$.

By definition:

$$T_p S^1 = \{[p, \varphi, v] \mid v \in \mathbb{R}\}$$

To see the intuitional picture we compute $T_p\iota : T_p S^1 \longrightarrow T_{\iota(p)}\mathbb{R}^2 \cong p + \mathbb{R}^2$

$$T_p\iota([p, \varphi, v]) = [\iota(p), id_{\mathbb{R}^2}, D(id_{\mathbb{R}^2} \circ \iota \circ \varphi^{-1})(\varphi(p))v]$$

$p = (x_0, y_0)$

$$D(id_{\mathbb{R}^2} \circ \iota \circ \varphi^{-1})(\varphi(p))v \qquad \Bigg| \quad \varphi(p) = x_0$$

$$= D\left(\begin{pmatrix} x \\ \sqrt{1-x^2} \end{pmatrix}\right)(x_0)\, v$$

$$= \begin{pmatrix} 1 \\ -\dfrac{x_0}{\sqrt{1-x_0^2}} \end{pmatrix} v = \begin{pmatrix} y_0 \\ -x_0 \end{pmatrix} \frac{v}{y_0}$$

$T_p\iota([p, \varphi, v])$ (3rd component)

(Note: $v \in \mathbb{R}$.)

For S^2 $T_p\iota$ maps T_pS^2 to

We identify T_pS^2 with $T_p\iota(T_pS^2)$

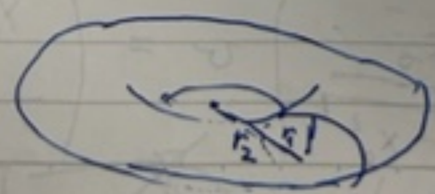
Convention 1.19: More general if $N \subseteq M$ is a smooth submanifold then we identify T_pN with $T_p\iota(T_pN)$ $(\subseteq T_pM)$

Example 1.20: Take $0 < r_1 < r_2$

$$M = \{(x,y,z) \in \mathbb{R}^3 \mid (\sqrt{x^2+y^2} - r_2)^2(x^2+y^2) + z^2(x^2+y^2) = r_1^2 \cdot (x^2+y^2)\} \setminus \{(0,0,z)\} \subseteq \mathbb{R}^3$$

M? Formula inside $\Leftrightarrow$

$$\left| (x,y,z) - \frac{r_2}{\sqrt{x^2+y^2}}(x,y,0) \right|_2 = r_1$$



M is a C^1- submanifold of $\mathbb{R}^3$.

Proof: We apply Problem 2 of Sheet 1.

$$F(x,y,z) := \left(\sqrt{x^2+y^2} - r_2\right)^2 (x^2+y^2) + z^2(x^2+y^2) - r_1^2(x^2+y^2).$$

$F \in C^1(\mathbb{R}^3 \setminus e_3\mathbb{R}, \mathbb{R})$.

$M = \{(x,y,z) \in \mathbb{R}^3 \mid F(x,y,z) = 0\}$

Take $P = (x_0, y_0, z_0) \in M$. We need to show that $DF(x_0, y_0, z_0) \neq (0,0,0)$.

Case $z_0 \neq 0$: Then $\frac{\partial F}{\partial z}(P) = 2z_0 \underbrace{(x_0^2+y_0^2)}_{\neq 0} \neq 0$. ✓

$\Rightarrow DF(P) \neq 0$

Case $z_0 = 0$ and $x_0 \neq 0$: Then $\frac{\partial F}{\partial x}(P)$

$$= (x_0^2 + y_0^2) \frac{\left(\sqrt{x_0^2+y_0^2} - r_2\right)}{\sqrt{x_0^2+y_0^2}} \cdot 2x_0$$

↑ Exercise (Note: $P \in M$)

$\neq 0$ because $x_0 \neq 0$ and $\left(\sqrt{x_0^2+y_0^2} - r_2\right)^2 = r_1^2$.

Case $z_0 = 0$ and $y_0 \neq 0$: Similar to the above case.

<u>Claim:</u> ~~∃~~ $M \overset{C^1}{\simeq} S^1 \times S^1$
(product manifold)

<u>Proof:</u>

- For that we find a better atlas for M.

We have

$$M = \{ (r_2 \cos\theta + r_1 \cos\alpha \cos\theta,\ r_2 \sin\theta + r_1 \cos\alpha \sin\theta,\ r_1 \sin\alpha) \mid \alpha, \theta \in [0, 2\pi[\}$$

$$= \{ P(\alpha, \theta) \mid \alpha, \theta \in [0, 2\pi[\}$$

We allow $\alpha, \theta \in]-\varepsilon, 2\pi + \varepsilon[$

The submanifold charts are given by projections to the coordinate planes. Thus by the Inverse function theorem it is enough to show that

$$(\alpha, \theta) \xmapsto{G} \begin{pmatrix} x(\alpha, \theta) \\ y(\alpha, \theta) \end{pmatrix}$$

in case of $z_0 \neq 0$

and

$$(\alpha, \theta) \longmapsto \begin{pmatrix} x(\alpha, \theta) \\ z(\alpha, \theta) \end{pmatrix}$$

in case of $z_0 \neq 0$ and $x_0 \neq 0$

and $(\alpha, \theta) \longmapsto \begin{pmatrix} Y(\alpha,\theta) \\ Z(\alpha,\theta) \end{pmatrix}$
in case of $z_0 = 0 = x_0$

have invertable derivative at (α_0, θ_0).
Ex. (Case 1)

$$DG(\alpha_0, \theta_0) = D\begin{pmatrix} X(\alpha,\theta) \\ Y(\alpha,\theta) \end{pmatrix}(\alpha_0, \theta_0)$$

$$= \begin{pmatrix} -\sin\alpha_0\, r_1 \cos\theta_0 & -(r_2 + r_1\cos\alpha_0)\sin\theta_0 \\ -\sin\alpha_0\, r_1 \sin\theta_0 & (r_2 + r_1\cos\alpha_0)\cos\theta_0 \end{pmatrix}$$

$$\det(\quad) = -\sin\alpha_0 \cdot (r_2 + r_1\cos\alpha_0) r_1$$

$$= -z_0 \sqrt{x_0^2 + y_0^2} \neq 0.$$

- We define $S^1 \times S^1 \xrightarrow{H} M$
  via $\cancel{(\alpha, \theta)} \longmapsto P(\alpha, \theta)$.
  $(e^{i\alpha}, e^{i\theta})$

H is bijective and C^1 and
$T_Q H$ is " $\forall Q \in S^1 \times S^1$, as verified

on charts above.

Inverse function theorem $\Rightarrow H^{-1}$ is C^1, so H is a diffeomorphism.
end 14.02.2023

We have a C^1-map $M \to S^1$ given by.

$$M \simeq S^1 \times S^1 \longrightarrow S^1$$
$$e^{i\alpha}, e^{i\theta} \longmapsto e^{i(\alpha+\theta)}$$

The second on charts: $(\alpha, \theta) \longmapsto (\alpha+\theta)$, therefore it is C^1.

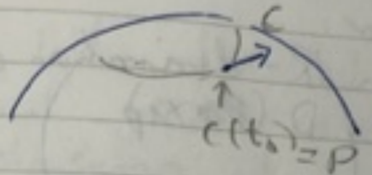
The interesting part of this map is the arrangement of the ~~fibst~~ fibers. (Hopf fibration)

Lecture 3

Remark 1.21: (a) The description of the tangent bundle of a smooth manifold M using an embedding into some $\mathbb{R}^k$ is extrinsic, because it depends on the embedding and on k.
But it suggests a new intrinsic description using C^1-curves, i.e. C^1-maps $I \to M$

Ex.: Take $\underset{(x_0, y_0, z_0)}{\overset{}{p}} \in S^2 \subseteq \mathbb{R}^3$

with $z_0 > 0$



$$c(t) := \left(\sqrt{x_0^2+y_0^2}\cos(t), \sqrt{x_0^2+y_0^2}\sin(t), z_0\right)$$

$$c'(t_0) = \begin{pmatrix} -\sqrt{x_0^2+y_0^2}\sin(t_0) \\ \sqrt{x_0^2+y_0^2}\cos(t_0) \\ 0 \end{pmatrix}$$

(B) $C^\infty(M,\mathbb{R})_p := \{[f,U]_p \mid U \subseteq M$ open, $p \in U$, $f \in C^\infty(U,\mathbb{R})\}$ "stalk of M at p"
$(f,U) \sim (g,V) \overset{def}{\Longleftrightarrow} \exists W \subseteq U \cap V$ open, $p \in W$: $f = g$ on W; $[f,U]_p$ "germ".

(C) A vector $v \in T_p\mathbb{R}^n$ defines a map $\partial_{v,p}: \underset{\text{"stalk at p"}}{C^\infty(\mathbb{R}^n, \mathbb{R})_p} \longrightarrow \mathbb{R}$

$$\partial_{v,p}(f) := \left.\frac{d(f(p+tv))}{dt}\right|_{t=0}.$$

It is · $\mathbb{R}$-linear

$$\partial_{v,p}(\lambda f) = \lambda\, \partial_{v,p}(f) \quad \forall_{\lambda \in \mathbb{R}}$$

$$\partial_{v,p}(f+g) = \partial_{v,p}(f) + \partial_{v,p}(g)$$

· satisfies the Leibniz rule

$$\partial_{v,p}(fg) = f(p)\,\partial_{v,p}(g) + g(p)\cdot\partial_{v,p}(f).$$

An $\mathbb{R}$-linear map satisf. LR is called "$\mathbb{R}$-derivation at p".

Every $\mathbb{R}$-derivation is given by a vector:

D an $\mathbb{R}$-derivation at p.

$$v := \begin{pmatrix} D(x_1) \\ \vdots \\ D(x_n) \end{pmatrix}$$

Then $D = \partial_{v,p}$.

Def 1.22: Let $(M, \mathcal{A}_r)$ be a smooth manifold. Take $p \in M$.

(a) A C^r-curve $c: I \longrightarrow M$ is said to start at p if $0 \in \mathring{I}$ and $c(0) = p$.

(b) Two C^1-curves $c: I \longrightarrow M$ and $d: J \longrightarrow M$ with $0 \in \mathring{I} \cap \mathring{J}$ are called Jet-equivalent if $c(0) = d(0)$ and $\exists\ (\varphi, U) \in \mathcal{A}_1$ around $c(0)$: $(\varphi \circ c)'(0) = (\varphi \circ d)'(0)$.

We denote the equivalence class by $[c]_1$.

We also write $c \overset{jet}{\sim} d$.

(c) Think about how one would define r-jet equivalence for curves.

<u>Remark 1.23:</u> Let $(M, \mathcal{A}_1)$ be C^1-manifold and $c: I \longrightarrow M$, $d: J \longrightarrow M$ C^1-curves starting at $p_0 \in M$.
Then are equivalent:
1° $[c]_1 = [d]_1$

2° $(T_0 c)([0, id_{I^\circ}, 1_{\mathbb{R}}]) = (T_0 d)([0, id_{J^\circ}, 1])$

<u>Proof:</u> $(T_0 c)([0, id_{I^\circ}, 1])$

$= [p_0, \varphi, D(\varphi \circ c \circ id_{I^\circ}^{-1})(id_{I^\circ}(0)) \cdot 1]$

$= [p_0, \varphi, (\varphi \circ c)'(0)]$ $\square$

We write $c'(0)$ for $(T_0 c)([0, id_{I^\circ}, 1_{\mathbb{R}}])$.

<u>Def 1.24:</u> Let $(M, \mathcal{A}_\infty)$ be a C^∞-manifold and $p \in M$.
The set
$$\mathrm{Der}_p(M) = \{D : \mathcal{C}^\infty(M, \mathbb{R})_p \to \mathbb{R} \mid D \text{ is } \mathbb{R}\text{-linear and satisfies the Leibniz rule at } p\}$$

is called ~~called~~ the set of derivations at p.

<u>Lemma 1.25:</u> Let M^n be a C^∞-manifold and (φ, U) be a C^∞-chart for M and $p \in U$.
Suppose $\varphi(p) = \underline{0} \in \mathbb{R}^n$.
Let $x_i : U \to \mathbb{R}$ be the i-th coordinate of φ, i.e. $\varphi = (x_1, \dots, x_n)$.

Then the map

$$\Phi : Der_p(M) \longrightarrow \mathbb{R}^n$$
$$D \longmapsto \begin{pmatrix} D(x_{1p}) \\ \vdots \\ D(x_{np}) \end{pmatrix}$$

is an $\mathbb{R}$-linear isomorphism.

<u>Proof:</u> · $\mathbb{R}$-linear ✓
· surjective: Take $v \in \mathbb{R}^n$.
Define $D_v \in Der_p(M)$ via $D_v(f_p) := \frac{d(f(\varphi^{-1}(tv}{dt}$

Then $\Phi(D_v) = \begin{pmatrix} \cancel{v_1} \\ \cancel{\vdots} \\ \cancel{v_n} \end{pmatrix} = v$

- injectivity: We show $\ker(\Phi) = 0$. the zero derivation.
Let $D \in \ker(\Phi)$, ~~and let [illegible]~~ i.e. we have $D(x_{1,p}) = \ldots = D(x_{n,p}) = 0$.
Take $f_p \in \mathcal{C}^\infty(M,\mathbb{R})_p$, the germ of some $f \in \mathcal{C}^\infty(M,\mathbb{R})$ at p.

We have $f(Q) = (f \circ \varphi^{-1})(x_1(Q), \ldots, x_n(Q))$

This way we get $Der_p(M) \simeq Der_{\underline{0}}(\mathcal{C}(U), \mathbb{R})$ ~~and it is enough to consider $x = [illegible]$~~.

We obtain

$$f(Q) - f(P) = (f \circ \varphi^{-1})(x_1(Q), \ldots, x_n(Q)) - (f \circ \varphi^{-1})(\underbrace{x_1(P), \ldots, x_n(P)}_{\underline{0}})$$

$$= \int_0^1 \frac{d(f \circ \varphi^{-1})(t x_1(Q), \ldots, t x_n(Q))}{dt} \, dt$$

$$= \sum_{j=1}^{n} x_j(Q) \underbrace{\int_0^1 \frac{\partial (f \circ \varphi^{-1})}{\partial x_j}(t x_1(Q), \ldots, t x_n(Q)) \, dt.}_{=: \, g_j(x_1(Q), \ldots, x_n(Q))}$$

Then $g_1, \dots, g_n \in C^\infty$ locally around p.

and
$$D(f_p) = \sum_{j=1}^{n} D((x_j g_j)_p)$$
$$= \sum_{j=1}^{n} \left(x_j(p)\, D(g_{j,p}) + g_j(p)\, D(x_{j,p}) \right)$$
$$\underset{x_j(p)=0}{=} \sum_{j=1}^{n} g_j(p) \underbrace{D(x_{j,p})}_{=0} = 0$$

Thus $D = 0$ □

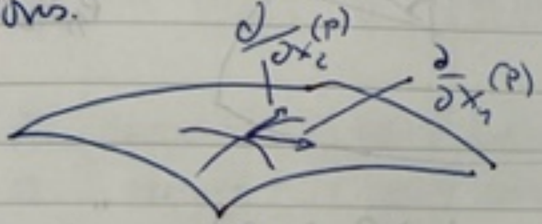
Prop 1.26: (a) Let M be C^1 manifold and $p \in M$. Then $T_pM \simeq \{[c]_1 \mid c \text{ a } C^1 \text{ curve on } M \text{ starting at } p\}$ canonically.

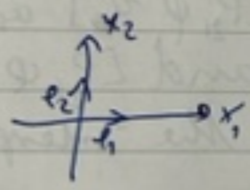
(b) Let M be a C^∞ manifold and $p \in M$. Then $T_pM \simeq \mathrm{Der}_p(M)$ canonically.

Proof: (a) We just give the map
$$T_pM \longrightarrow \{[c]_1 / \dots\}$$
$$[p, \varphi, v] \longmapsto [t \mapsto \varphi^{-1}(\varphi(p) + tv)]_1$$
$$[p, \varphi, (\varphi \circ c)'(0)] \longleftarrow [c]_1$$

(b) Exercise. □

<u>Notation 1.27:</u> The tangent vectors
$[p, \varphi, e_1], \dots, [p, \varphi, e_n]$ (w.r.t. $(\varphi = (x_1, \dots, x_n), U)$)
are denoted by
$\frac{\partial}{\partial x_1}(p), \dots, \frac{\partial}{\partial x_n}(p)$
to also emphasize the description as derivations.





For a C^r- $(r>0)$ manifold M^n we can find a C^{r-1} differentiable structure on TM with charts

$$T\varphi : TU \longrightarrow \varphi(U) \times \mathbb{R}^n \quad \text{for } (\varphi, U) \in \mathcal{A}^r.$$
$$[p, \varphi, v] \longmapsto (\varphi(p), v)$$

<u>Check:</u> · The TU cover TM.
· For $(\varphi, U), (\psi, V) \in \mathcal{A}^r$ we have
$$(T\varphi)(T\psi)^{-1}(\psi(p), w) = (\varphi(p), D(\varphi\psi^{-1})(\psi(p))w)$$
Thus $(T\varphi)(T\psi)^{-1} \in C^{r-1}(\psi(V \cap U) \times \mathbb{R}^n, \varphi(V \cap U) \times \mathbb{R}^n)$

17.02.2023