INTRODUCTION TO MATHEMATICAL FINANCE
PROBLEM SHEET 10

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Problem 1 (10, time value). Let $((\underline{S}_t)_{t=\overline{0,T}}, (\mathcal{F}_t)_{t=\overline{0,T}}$ in $\mathcal{F}, \mathrm{P})$ be a 1-period market model with

(i) constant spot rate $r \in (-1, \infty)$ and $S_0^0 = 1$,
(ii) one risky asset $S = S^{(1)}$ with possible trajectories
$$\{(S_0(\omega), S_1(\omega)) |\ \omega \in \Omega\} = \{(1, \frac{1}{2}), (1, 2)\},$$
i.e. those trajectories are exactly the ones with positive probability,
(iii) $\mathcal{F}_0 = \{\varnothing, \Omega\},\ \mathcal{F}_1 = \sigma(S^{(1)})$.

We consider for every $\lambda > 0$ a new asset:
$$S_t^{(\lambda)} := \lambda S_t^{(1)},\ t = 0, 1,$$
and the put option C_λ^{put} on $S^{(\lambda)}$ with strike 1. Let us assume that the market model is arbitrage-free.

(i) Show that the function $\pi_r : (0, \infty) \to (0, \infty)$ which maps λ to the arbitrage-free price of C_λ^{put} is convex.
(ii) For which r there is a λ such that the put C_λ^{put} has a negative time value?
(iii) Suppose the spot rate is equal to $\frac{1}{2}$. For which λ the put option C_λ^{put} has a vanishing time value?

Problem 2 (10, completeness and extremal points). Let $((\underline{S}_t)_{t=\overline{0,T}}, (\mathcal{F}_t)_{t=\overline{0,T}}$ in $\mathcal{F}, \mathrm{P})$ be a multi-period market model such that $\mathcal{F} = \mathcal{F}_T$ and F_0 is the trivial σ-algebra. Let $\mathcal{Q}$ be the set of martingale measures for the discounted assets. We define for $\mathrm{Q} \in \mathcal{Q}$ the set
$$\mathrm{A}_\mathrm{Q} := \{\tilde{Q} \in \mathcal{Q} |\ \tilde{Q} << \mathrm{Q}\}.$$
Prove or disprove for an element $\mathrm{Q} \in \mathcal{Q}$ the equivalence of the following two assertions:

(i) $\mathrm{A}_\mathrm{Q} = \{\mathrm{Q}\}$.
(ii) $\mathrm{Q} \in \mathrm{Ext}(\mathcal{Q})$.

Problem 3 (10, set of martingale measures). Let $((\underline{S}_t)_{t=\overline{0,T}}, (\mathcal{F}_t)_{t=\overline{0,T}}$ in $\mathcal{F}, \mathrm{P})$ be a two-period market model such that

- $d = 1$, i.e. we have one bond and one risky asset S,
- $\mathcal{F} = \mathcal{F}_2$ and $\mathcal{F}_t = \sigma(S_1, \dots, S_t),\ t = 0, 1, 2$,
- $S_t^{(0)} = 1$, $t = 0, 1, 2$,
- the asset S has the following possible trajectories
$$(\frac{7}{2}, 1, \frac{1}{2}), (\frac{7}{2}, 1, 2), (\frac{7}{2}, 3, 1), (\frac{7}{2}, 3, 4), (\frac{7}{2}, 4, 3), (\frac{7}{2}, 4, 5).$$

(i) Compute the set of martingale measures for S.
(ii) Compute the set of martingale measures for the market model.
(iii) Visualize your results in $\mathbb{R}^3$.

Problem 4 (10, bank with extra information). Suppose in $((\underline{S}_t)_{t=\overline{0,T}}, (\mathcal{F}_t)_{t=\overline{0,T}}$ in $\mathcal{F}, \mathrm{P})$, a complete market with trivial σ-algebra $\mathcal{F}_0$, there is a bank with more information, more precisely there is a filtration $(\mathcal{G}_t)_{t=\overline{0,T}}$ in $\mathcal{F} = \mathcal{F}_T$, which satisfies

Date: Please hand in before the lecture by **12th of May 2022**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference.

- $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{F}_{t+1}$ for every $t < T$,
- and there exists a time $s < T$ and an index $i \geqslant 1$ such that
$$\mathrm{E}^*[\mathrm{X}_{s+1}^{(i)}|\mathcal{F}_s] \neq \mathrm{E}^*[\mathrm{X}_{s+1}^{(i)}|\mathcal{G}_s]$$
  for the ith asset with respect to the unique martingale measure P^*.

Show that the bank has an arbitrage opportunity, and explain how you get a realizing trading strategy in terms of $(\mathcal{G}_t)_{t=\overline{0,T}}$ and completeness.