Mathematical finance I

0. Interlude: Measure theory

measure space $(\Omega, \mathcal{F}, \mu)$, $\Omega \neq \emptyset$, $\mathcal{F}$ σ-algebra

$\mu : \mathcal{F} \longrightarrow \mathbb{R}_+ \cup \{\infty\}$

is a measure.

Def 1: μ „σ – endlich", if

$\exists\, (A_n)_{\mathbb{N}} : A_n \in \mathcal{F}$ and $\Omega = \cup A_n$

and $\mu(A_n) < \infty$

Ex 2: $\Omega = \mathbb{R}^d$, $\mathcal{F} = \mathcal{B}(\mathbb{R}^d)$, $\mu = \lambda \big|_{\mathcal{B}(\mathbb{R}^d)}$

(the Borel measure)

Rem 3: If $|\Omega| \leq |\mathbb{N}|$ then if $\mathcal{F} = \mathcal{P}(\Omega)$:

μ is σ-finite $\Longleftrightarrow$ $\forall \omega \in \Omega$: $\mu(\{\omega\}) < \infty$

(exercise)

Def 4: μ is called absolutely continuous w.r.t. ν

($(\Omega, \mathcal{F}, \mu)$, $(\Omega, \mathcal{F}, \nu)$ measured spaces), we write $\mu \ll \nu$,

if $\forall A \in \mathcal{F}$: $\nu(A) = 0 \Rightarrow \mu(A) = 0$

Prop 5: (Radon – Nikodym) Let μ, ν be σ-finite; then are equivalent.

(1) $\nu \ll \mu$

(2) $\exists f : \Omega \to \mathbb{R}_+ \cup \{\infty\}$ $\mathcal{F}$ measurable s.t. $\forall A \in \mathcal{F}$: $\nu(A) = \int_A f \, d\mu$.

Terminology 6: f is called the Radon – Nikodym density $f =: \frac{d\nu}{d\mu}$.

f is ~~unix~~ μ- almost surely unique.

Example 7: $\Omega = \mathbb{R}_+$, $\mathcal{F} = \mathcal{B}(\mathbb{R})$ $\mu = \lambda|_{\mathcal{B}(\mathbb{R})}$

~~F~~ Let ν be a measure on $\mathcal{F}$. We put

$F_\nu(x) := \nu([0, x])$ (Why is this enough to know ν?)

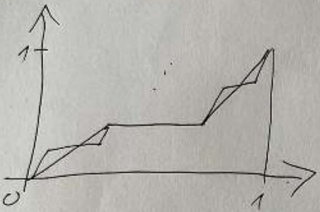
If $\nu \ll \mu$ and im $F_\nu \subseteq \mathbb{R}$ then $\exists f : \mathbb{R}_+ \to \mathbb{R}$:

$$\nu(A) = \int_A f \, dx \quad , A \in \mathcal{B}([0, \infty[)$$

We have $F_\nu'(x) = f(x)$ where F_ν is differentiable. (almost everywhere.

Be careful: We have the following example

$\Omega = [0,1]$ $\quad F_\nu :=$ Cantor's step function $: [0,1] \longrightarrow [0,1]$



Properties: • F_ν continuous

• $F_\nu' = 0$ almost everywhere.

• F_ν increasing

Define $\nu_F([0,x]) := F_\nu(x)$.

Then $\nu \not\ll \lambda|_{\mathcal{B}([0,1])}$, Because

$$f := 0 = \frac{dF_\nu}{dx} \quad \text{and} \quad \int_0^1 f(x)\,dx = 0 \neq F(1) = 1$$

Reason: There is a λ-zero set (Cantor set) C s.t. $\nu(C) = 1 \neq \lambda(C) = 0$.

We define $\nu \approx \mu$ if $\nu \ll \mu$ and $\mu \ll \nu$.

I. Arbitrage theory for the 1-period model

I.1 Asset, portfolio, arbitrage

Objects: market model. $d+1$ assets

- savings
- stocks
- option (derivatives)

1- period model: $t = 0$ and 1.

at time 0: $\pi^{(i)} \geq 0$ price for the ith asset.

$i=0$: e.g. Currency (as a ref.) or bond

* For simplicity we suppose $\pi^{(0)} = 1$.

Price vector $\underline{\pi} = (\pi^{(0)}, \dots, \pi^{(d)})$

at time 1: $S^{(i)} \geq 0$ random variable. Given $(\Omega, \mathcal{F}, P)$ a probability space

* We assume $S^{(0)} > 0$. almost surely (risk freeness for $S^{(0)}$.)

We put $r(\omega) = S^0(\omega) - \pi^0(\omega) = S^0(\omega) - 1 > -1$

"return" $\qquad S^0(\omega) > 0$

$\underline{S}(\omega) = (S^0(\omega), \dots, S^d(\omega))$

Portfolio: $\xi^i :=$ share of the ith asset in the portfolio.

$t=0$: value of — " — : $\sum \xi^i \pi^{(i)} = \underline{\xi} \cdot \underline{\pi}$

$t=1$: — " — : $\sum \xi^i S^{(i)} = \underline{\xi} \cdot \underline{S}$

sign of ξ^i:
- 0: nothing
- $-$ loan. ("Leerverkauf" "short sale")
- $+$ possession

$\underline{\xi}(\underline{S} - \underline{\pi})$ = gain, earnings, profit.

We assume $r(\omega) \geq 0$. (Otherwise we would consider negative interest rates.)

Def 8: Let $(\underline{\pi}, \underline{S})$ be a market model

A portfolio $\underline{\xi} = (\xi^0, \dots, \xi^d)$ is called an arbitrage opportunity (AO) if

(i) $\underline{\pi}\, \underline{\xi} \leq 0$

(ii) $\underline{S}\, \underline{\xi} \geq 0$ P-a.s.

(iii) $P(\underline{S}\, \underline{\xi} > 0) > 0$

Basic principle: $\nexists$ an arbitrage opportunity.

(we call this arbitrage free)

<u>Remark 9:</u> 1) If $\pi^{(i)} = 0$, and A.F

then $S^{(i)} = 0$ P-a.s.

2) The definition of A.F. is independent of the 0th asset, but depends on P.

<u>Notation 10:</u> $\underline{\xi} = (\xi_0, \dots, \xi_d)$

$\xi = (\xi_1, \dots, \xi_d)$. Same for $\underline{S}$, $\underline{\pi}$.

literature

<u>Lemma 11:</u> (1.3) Given a market model (MM)

the following assertions are <u>equivalent</u>.

1° MM has an arbitrage opportunity

2° $\exists\ \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$:

$$\xi \cdot S(\omega) = \sum \xi_i S^{(i)}(\omega) \geq (1 + r(\omega))\, \xi \cdot \pi$$

P-almost surely.

ξ and $P(\xi S(\omega) > (1+r(\omega))\xi\pi) > 0$.

end of lecture 1 15.02.22

Proof: $2^\circ) \Rightarrow 1^\circ$ Take $(-\xi\pi, \pi)$!

$1^\circ \Rightarrow 2^\circ$: Let $\underline{\xi}$ be an arbitrage opportunity for $(\Pi, \underline{S})$.

Then $\xi S(\omega) \geq -\xi^0 S^0(\omega) = -\xi^0(1+r(\omega))$

$$\underset{\uparrow}{=} \xi\pi(1+r(\omega))$$

$$-\xi^0 \geq \xi\pi$$

and $>$ with positive probability. $\square$

Remark 12: We write $Y^{(i)}(\omega) := \dfrac{S^{(i)}(\omega)}{1+r(\omega)} - \pi^{(i)}$

for discounted net gain of the i-th asset.

Then Lemma 11 states:

MM is A.F. $\Leftrightarrow$ ($Y \cdot \xi \geq 0$ P-almost surely $\Rightarrow$ $Y\xi = 0$ P-a.s.).

I.2 Arbitrage freeness and Martingale measures

We are given a MM $(\underline{\Pi}, \underline{S}, P)$.

<u>Def 13:</u> A measure P^* on $(\Omega, \mathcal{F})$ is called "risk-neutral" or "Martingale measure" if

$$\pi^{(i)} = E^*\left[\frac{S^{(i)}}{(1+r)}\right] \quad i = 1, \dots, d$$

E^* ↑ expectation w.r.t. P^*.

$$\mathcal{P} := \{ P^* \mid P^* \approx P \text{ and } P^* \text{ is a martingale measure} \}$$

<u>Remark 14:</u> Typical is $\pi^{(i)} < E_P\left[\frac{S^{(i)}}{1+r}\right]$,

i.e. one considers the risk more than the gain. So one pays a smaller preis than $E_P\left[\frac{S^{(i)}}{1+r}\right]$.

Related topic: expected utility. (maybe later)

<u>Proposition 15:</u> (1.6.) FTAP "Fundamental Theorem of Asset Pricing"

A market model $(\underline{\Pi}, \underline{S}, P)$ is AF iff $\mathcal{P} \neq \emptyset$.

Further, if $\mathcal{P} \neq \emptyset$ then $\exists\, P^* \in \mathcal{P}: \frac{dP^*}{dP}$ ~~$\frac{dP}{dP^*}$~~ is bounded.

Proof: "$\Leftarrow$" We have $\mathbb{P}^* \neq \emptyset$. Assume $(\underline{\Pi}, \underline{S}, P)$ is not AF. Then $\exists\, \underline{\xi} \in \mathbb{R}^{d+1}$:

- $\underline{\xi}\,\underline{\Pi} \leq 0$
- $\underline{\xi}\,\underline{S} \geq 0$ P-a.a.
- $P(\underline{\xi}\,\underline{S} > 0) > 0$.

This also holds for P^*, because $P^* \approx P$.

$$\Rightarrow\ 0 < E^*\Big[\underline{\xi}\,\underline{S}\,\frac{1}{1+r}\Big] = \sum_{i=0}^{d} \xi^{(i)}\, E^*\Big[\frac{S^{(i)}}{1+r}\Big]$$

$$\underset{\substack{\uparrow \\ P^* \text{ martingale} \\ \text{measure}}}{=} \underline{\xi}\,\underline{\Pi} \leq 0 \qquad \lightning$$

"$\Rightarrow$" We have that $(\underline{\Pi}, \underline{S}, P)$ is AF.

To show $\mathbb{P} \neq \emptyset$.

Step 1: We consider at first the case $E[|Y^{(i)}|] < \infty$, $i = 1, \dots, d$.

$$\mathcal{Q} := \Big\{ Q \mid Q \approx P \text{ and } \frac{dQ}{dP} \text{ is bounded} \Big\}$$

$\mathcal{Q}$ is convex, i.e. $\forall\, Q_1, Q_2 \in \mathcal{Q}\ \forall\, \lambda \in [0,1]$:

$\lambda Q_1 + (1-\lambda) Q_2 \in \mathcal{Q}$. (exercise)

$$\mathcal{C} := \big\{ E_Q[Y] \mid Q \in \mathcal{Q} \big\} \subseteq \mathbb{R}^d$$

$$\left(E_Q[Y] = \begin{pmatrix} E_Q[Y^{(1)}] \\ \vdots \\ E_Q[Y^{(d)}] \end{pmatrix} \right)$$

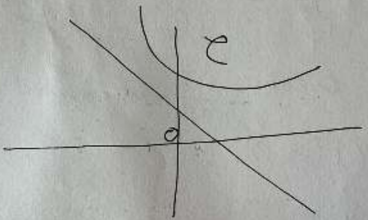
Note that $E_Q[Y]$ is well-defined, because $\frac{dQ}{dP}$ is bounded.

$\mathcal{C}$ is convex, because $\mathcal{Q}$ is convex.

<u>To show:</u> $0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathcal{C}$.

Assume $0 \notin \mathcal{C}$.

Then $\exists$ hyperplane H such that 0 and $\mathcal{C}$ are on different sides of the hyperplane,



i.e. $\underset{\substack{0 \neq \xi \perp H \\ \in \mathbb{R}^d}}{\exists} \ \exists\, k \geq 0$:
- $H = \{ x \in \mathbb{R}^d \mid \xi x = k \}$
- $0 \in \{ \quad \mid \xi x \leq k \}$
- $\mathcal{C} \subseteq \{ \quad \mid \xi x \geq k \}$

and $\exists\, x_0 \in \mathcal{C}$: $\xi x_0 > k$.

We just can take $k = 0$.

$\underset{(I)}{\Rightarrow} \left(\forall_{Q \in \mathcal{Q}} : E_Q[\xi Y] \geq 0 \right)$ and

$(II) \left(\exists_{Q_0 \in \mathcal{Q}} : E_{Q_0}[\xi Y] > 0 \right)$

$(II) \Rightarrow \quad P(\xi Y > 0) > 0$

<u>Claim:</u> (I) implies $\xi Y \geq 0$ P. a.s..

<u>Proof</u> (claim) $A := \{ \omega \in \Omega \mid \xi Y(\omega) < 0 \}$

Define $\varphi_n(\omega) := (1 - \frac{1}{n}) \mathbb{1}_A(\omega) + \frac{1}{n} \mathbb{1}_{A^c}(\omega)$. and consider the measure

$$Q_n \text{ given by } \frac{dQ_n}{dP} := \varphi_n \frac{1}{E_P[\varphi_n]}.$$

$\Rightarrow Q_n \in \mathcal{Q}$, because $\frac{dQ_n}{dP}$ is bounded.

$$\overset{I}{\Rightarrow} \quad E_{Q_n}[\xi Y] \geq 0$$

$$\| $$

$$\frac{E_P[\xi Y \cdot \varphi_n]}{E_P[\varphi_n]}$$

$\Rightarrow 0 \leq E_P[\xi Y \varphi_n] \xrightarrow[\text{bounded convergence}]{n \to \infty} E_P[\xi Y \mathbb{1}_A]$

$\Rightarrow P(A) = 0$. $\square$ (claim)

⌈Recap: bounded convergence: $X_n \to X$ μ.a.s. and $|X_n| \leq Y \in L^1(\mu)$, then $E[X_n] \to E[X]$

Thus $(-\xi\pi, \xi)$ is an AO. ↯ So $0 \in \mathcal{C}$.

<u>Step 2:</u> $E_P[|Y^{(i)}|] = \infty$ for some i.

<u>Idea:</u> Change P to $\tilde{P}$ such that $E_{\tilde{P}}[|Y^{(i)}|] < \infty$.

Put $\dfrac{d\tilde{P}}{dP} = \dfrac{c}{1+|Y|_2}$. Then

$$E_{\tilde{P}}[|Y|_2] = E_P\left[\frac{|Y|_2}{1+|Y|_2} \cdot c\right]$$

$$\leq c < \infty$$

$$\Rightarrow E_{\tilde{P}}[|Y^{(i)}|] \leq E_{\tilde{P}}[|Y|_2] < +\infty.$$

We have $\tilde{P} \approx P$ and $\left|\dfrac{d\tilde{P}}{dP}\right| \leq c$. $\square$

<u>Example 16:</u> $\Omega = \{\omega_1, \omega_2, \dots, \omega_{\underset{\geq 2}{N}}\}$, $\mathcal{F} = \mathcal{P}(\Omega)$

Given $(\Pi, \underline{S}, P)$ s.t. r constant and $d = 1$ and $P(\{\omega_j\})$

Put $s_j := S^{(1)}(\omega_j)$, $j = 1, \dots, N$.

Condition for AF? AF $\Leftrightarrow \mathcal{P} \neq \emptyset$

$$\mathcal{P} = \Big\{ Q \;\Big|\; q_j := Q(\{\omega_j\}) > 0,\ j = 1, \dots, N,\ \sum_{j=1}^{N} q_j = 1 \text{ and } E_Q\left[\frac{1}{1+r} S^{(1)}\right] = \pi^{(1)} \Big\}$$

$$\begin{cases} \dfrac{1}{1+r} \displaystyle\sum_{j=1}^{N} q_j s_j = \pi^{(1)} \\ \displaystyle\sum_{j=1}^{N} q_j = 1 \\ q_j \gneqq 0, \quad j = 1, \dots, N \end{cases}$$

Suppose $s_1, \dots, s_N$ are pairwise different.

We have $\mathcal{P} \neq \emptyset$ iff $(1+r)\pi^{(1)} \in]\min\{s_1, \dots, s_N\}, \max\{s_1, \dots, s_N\}[$.

and uniqueness iff $N = 2$.

end of Lecture 2 17.2.22

Def 17: $\mathcal{V} := \{\underline{\xi}\,\underline{S} \mid \underline{\xi} \in \mathbb{R}^{d+1}\} \subseteq \mathcal{L}^0(P)$

"space of attainable payoffs".

Def 18: A market model is called redundant if $\dim \mathcal{V} = d+1$ P-a.s. ; i.e. $S^{(0)}, \dots, S^{(d)}$ are $\mathbb{R}$-linearly independent in $\mathcal{L}^0(P)$.

$$\Leftrightarrow \forall_{\underline{\xi} \in \mathbb{R}^{d+1}} \left(\underline{\xi}\,\underline{S}(\omega) = 0 \ P\text{-a.s.} \Rightarrow \underline{\xi} = \underline{0} \right)$$

We could make a similar definition for attainable net gains. But we have the following proposition.

<u>Prop 19:</u> (a) Suppose a MM $(\underline{\Pi}, \underline{S}, P)$ is not redundant. Then $Y^{(1)}, \dots, Y^{(d)}$ are $\mathbb{R}$-linearly independent in $L^0(P)$.

(b) Suppose $Y^{(1)}, \dots, Y^{(d)}$ are $\mathbb{R}$-linearly independent in $L^0(P)$ and that $(\underline{\Pi}, \underline{S}, P)$ is AP. Then $(\underline{\Pi}, \underline{S}, P)$ is ~~not~~ redundant.
not

<u>Proof:</u> (a) $\xi Y = 0$ P-a.s.

$$\Rightarrow \quad \xi S - (1+r)\cdot \xi \pi = 0 \quad P\text{-a.s.}$$

$$\Rightarrow \quad \xi = 0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \text{ and } \xi \pi = 0$$

$S^{(1)}, \dots, S^{(d)}$ are $\mathbb{R}$-lin. ind in $L^0(P)$

$$\Rightarrow \quad \xi^{(1)} = \dots = \xi^{(d)} = 0.$$

(b) $\xi S + \xi^0 (1+r) = 0$ P-a.s.

$$\Rightarrow \quad \xi Y + \underline{\xi}\,\underline{\Pi} = 0 \qquad P\text{-a.s.} \quad (*)$$

$$\underset{P^* \in \mathcal{P}}{\Rightarrow} \quad 0 = E^*[\xi Y + \underline{\xi}\,\underline{\Pi}] \underset{\substack{P^* \text{ martingale} \\ \text{measure}}}{=} 0 + \underline{\xi}\,\underline{\Pi}$$

$$\underset{(*)}{\Rightarrow} \quad \xi Y = 0 \quad P \text{ a.s.}$$

$\Rightarrow \quad \xi = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$, because $Y^{(1)}, \dots, Y^{(d)}$ $\mathbb{R}$-linearly ind. in $L^0(P)$.

<u>Lemma 20</u> (1.11) Suppose $(\underline{\Pi}, \underline{S}, P)$ is AF.

Let $V \in \mathcal{V}$ and $\underline{\xi}, \underline{\zeta} \in \mathbb{R}^{d+1}$ s.t.

$\underline{\xi}\,\underline{S} = V = \underline{\zeta}\,\underline{S}$. Then $\underline{\xi}\,\underline{\Pi} = \underline{\zeta}\,\underline{\Pi}$.

(We cannot get a better price if we attain differently.)

I.3 Derivatives

Derivatives (also called
- derivative securities
- options
- contingent claims)

are securities which depend on $\underline{S}$ in a non-linear way.
How do we price such a security?

Examples 20:

1) "forward contract"
The agent sells at $t=1$ an asset to a price K.

$$C^{fw} = S^{(i)} - K.$$

3 scenarios:
- If $S^{(i)} > K$ then we have a gain
- If $S^{(i)} < K$ — " — loss
- If $S = K$, no gain, nor loss.

2) "call option"
The holder has the right to buy an asset for a price K at time $t=1$.
(but he is not obliged to buy.)

$$C^{call} = (S^{(i)} - K)^{+} := \begin{cases} S^{(i)} - K, & S^{(i)} \geq K \\ 0, & S^{(i)} < K \end{cases}$$

 K is called the "strike price"

3) "put option"

The holder is allowed to sell an asset for a price K at $t=1$.

$$C^{put} := (K - S^{(1)})_+$$

We have the put-call-parity:

$$C^{call} - C^{put} = S^{(1)} - K = C^{fw}.$$

For the discounted price we get

$$\pi(C^{fw}) = \pi(C^{call}) - \pi(C^{put})$$

$\Big(\pi(C) := \cancel{E^*\left[\frac{C}{1+r}\right]}\ \cancel{\int P^* \, C \, dP} \Big)$ price of discounted C.)

~~price of discounte~~

4) "basket index option"

$$V := \underline{\xi}\,\underline{S}$$

$C^{call} = (V - K)$ call option on a portfolio.

5) "straddle"

$$C = (\pi(V) - V)_+ + (V - \pi(V))_+$$
$$= |\pi(V) - V|$$

Def. 21: A random variable $C : \Omega \to \mathbb{R}_+$ is called <u>contingent claim</u>, if

C is $\sigma(S^{(0)}, \dots, S^{(d)})$ measurable, i.e.

$\exists\, f : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}_+$ Borel measurable

$\left(f^{-1}(B) \in \mathcal{B}(\mathbb{R}^{d+1})\ \forall B \in \mathcal{B}(\mathbb{R}_+) \right)$

such that $C(\omega) = f(S^{(0)}(\omega), S^{(1)}(\omega), \dots, S^{(d)}(\omega))$

$\forall \omega \in \Omega$. (exercise)

<u>Setup 22:</u> $(\underline{\Pi}, \underline{S}, P)$ a MM., C a contingent claim $C = f(S^{(0)}, \dots, S^{(d)})$.

Then we get a new MM:

$$((\underline{\Pi}, \pi^C), (\underline{S}, C), P)$$

for each $\pi^C \in \mathbb{R}_+$. Call it MM_{π^C}

<u>Def 23:</u> π^C is called an AF price, if MM_{π^C} is AF.

Put $\Pi(C) = \{ \pi^C \in \mathbb{R}_+ \mid MM_{\pi^C} \text{ is A.F.} \}$

Prop (1.30) 24: Suppose $\mathcal{P} \neq \emptyset$, i.e. $(\Pi, \underline{S}, P)$ is AF.
And suppose C is a contingent claim for MM. Then

$$\emptyset \neq \Pi(C) = \left\{ E^*\left[\frac{C}{1+r}\right] \,\middle|\, \begin{array}{l} P^* \in \mathcal{P} \\ E^*\left[\frac{C}{1+r}\right] < \infty \end{array} \right\}.$$

(In particular the price is not unique in general, because C does not need to depend linearly on $\underline{S}$.)

Proof: Denote the right set by R.
To show $\emptyset \neq \Pi(C) = R$.

$\Pi(C) \subseteq R$: $\pi^c \in \Pi(C)$. From $\mathcal{P}_{MM_{\pi^c}} \subseteq \mathcal{P}$ follows $\pi^c \in R$.

$R \subseteq \Pi(C)$: $\pi^c \in R \Rightarrow \exists\, P^* \in \mathcal{P} : \pi_c = E^*\left[\frac{C}{1+r}\right]$.

$$\Rightarrow P^* \in \mathcal{P}_{MM_{\pi^c}} \Rightarrow \pi^c \in \Pi(C)$$

$\Pi(C) \neq \emptyset$: Take $\tilde{P} \approx P$ such that $\tilde{E}\left[\frac{C}{1+r}\right] < \infty$
(FTAP) $\Rightarrow \exists\, P^* \in \mathcal{P}$ with $\frac{dP^*}{d\tilde{P}}$ bounded.
Thus $E^*\left[\frac{C}{1+r}\right] < \infty \Rightarrow R \neq \emptyset$ $\square$

end of Lecture 3. 22.02.22

We write $\Pi_{\inf}(C) := \inf \Pi(C)$

$\Pi_{\sup}(C) := \sup \Pi(C)$.

Prop 25: Let $(\underline{\Pi}, \underline{S}, P)$ be an AFMM and C be a contingent claim. Then

(a) $$\Pi_{\inf}(C) = \inf_{P^* \in \mathcal{P}} E^*\left[\frac{C}{1+r}\right]$$
$$= \max\left\{ m \in [0, \infty[\;\middle|\; \exists \xi \in \mathbb{R}^d : m + \xi Y \leq \frac{C}{1+r} \quad P\text{-a.s.} \right\}$$

(b) $$\Pi_{\sup}(C) = \sup_{P^* \in \mathcal{P}} E^*\left[\frac{C}{1+r}\right]$$
$$= \min\left\{ m \in [0, \infty] \;\middle|\; \exists \xi \in \mathbb{R}^d : m + \xi Y \geq \frac{C}{1+r} \quad P\text{-a.s.} \right\}$$

Remark 26: 1) $\Pi_{\sup}(C)$ is the smallest price of a portfolio $\underline{\xi}$ s.t. $\underline{\xi}\,\underline{S} \geq C$ (super replication) "super hedging strategy" — The seller is secure against big payoffs of C.

$\Pi_{\inf}(C)$ is the biggest price of a portfolio $\underline{\xi}$ s.t. $\underline{\xi}\,\underline{S} \leq C$ (sub replication) — The buyer is safe against small payoffs of C.

Def 27: A contingent claim C is called "attainable" (or "replicable") if $\exists\ \underline{\xi} \in \mathbb{R}^{d+1}: C = \underline{\xi}\,\underline{S}$ P-a.s.
$\underline{\xi}$ is called "replicating portfolio".

Proposition 28: (1.34) Let $(\underline{\Pi}, \underline{S}, P)$ be AF and C be a contingent claim.
(a) C is replicable $\Longleftrightarrow$ $|\Pi(C)| = 1$
(b) C is not replicable $\Leftrightarrow$ $\pi_{\inf}(C) < \pi_{\sup}(C)$
Further, in case (b) we have $\Pi(C) =]\pi_{\inf}(C), \pi_{\sup}(C)[$, i.e. $\Pi(C)$ is an open intervall.

Proof of Prop 25: (a) By Prop 24 we only need to prove the second equality.
For $m \in [0, \infty[$ with $\exists\ \xi \in \mathbb{R}^d$: $m + \xi Y \leq \frac{C}{1+r}$ P-a.s.
we have for $P^* \in \mathcal{P}$: $m + 0 = m + E^*[\xi Y] \leq E^*[\frac{C}{1+r}]$.
This implies "$\geq$" for sup instead of max.
For "$\leq$": There is nothing to show if $\pi_{\inf}(C) = 0$.
Suppose $\pi_{\inf}(C) > 0$. Take $0 < \lambda < \pi_{\inf}(C)$.
Then $((\underline{\Pi}, \lambda), (\underline{S}, C), P)$ has an AO by FTAP
(because $\mathcal{P}_{MM_{\pi_C = \lambda}} = \emptyset$)

Thus $\exists\, \xi \in \mathbb{R}^d$: $\sum_{i=1}^{d} \xi^{(i)} \left(\frac{S^{(i)}}{1+r} - \pi^{(i)} \right)$

$$+ 1 \cdot \left(\frac{C}{1+r} - \lambda \right) \geq 0 \text{ P-a.s.}$$

and > 0 with non-zero probability.

(Here we have used that (Π, S, P) has no A.O.)
and $\lambda < \pi_{\inf}(C)$.

Thus $\lambda - \xi Y \leq \frac{C}{1+r}$ P-a.s and

Therefor $\sup\{ m \in [0,\infty[\mid \exists\, \xi \in \mathbb{R}^d : \xi Y + m \leq \frac{C}{1+r} \}$
$\geq \lambda$.

λ arbitrary $\Rightarrow$ $\sup\{\} \geq \pi_{\inf}(C)$

Thus $\sup\{\dots\} = \pi_{\inf}(C)$.

We still need to show $\sup\{\dots\} = \max\{\dots\}$.

So let us take $\lambda_n \nearrow \pi_{\inf}(C)$. with $\xi_n \in \mathbb{R}^d$

s.t. $\lambda_n + \xi_n Y \leq \frac{C}{1+r}$ P-a.s. for all $n \in \mathbb{N}$.

Case 1: $\liminf_{n\to\infty} |\xi_n|_\infty < \infty$.

Then $\exists (n_j)_j$, $n_j \nearrow \infty$ $\exists\, \xi \in \mathbb{R}^d$: $\xi_{n_j} \xrightarrow{j\to\infty} \xi$

$\Rightarrow$ $\pi_{\inf} - \xi Y \leq \frac{C}{1+r}$ P-a.s.

Case II: $\liminf_{n\to\infty} |\xi_n|_\infty = \infty$.

Then $|\xi_n| \longrightarrow +\infty$

$\{x \in \mathbb{R}^d \mid |x|_\infty = 1\}$ is compact.

$\Rightarrow \exists\, n_j \uparrow \infty$: $\eta_{n_j} := \dfrac{\xi_{n_j}}{|\xi_{n_j}|_\infty}$ converges,

say to η.

$\Longrightarrow \quad \eta Y \leqq 0 \quad P$-a.s.

$\underset{\text{A.F.}}{\overset{}{\Longrightarrow}} \quad \eta Y = 0 \quad P$-a.s.

W.l.o.g. we could have assumed that $\underline{S}$ is not redundant.

$\Rightarrow \quad \eta = 0$ ~~P-a.s.~~ $\lightning$ because $|\eta|_\infty = 1$.

(b) is similar (exercise) $\square$

Proof of Proposition 28

(a) "$\Rightarrow$" The price of attainable payoffs is unique, thus so it is for replicable contingent claims.

"$\Leftarrow$" Let $\Pi(C) = \{\pi^C\}$

Prop. 25 $\Rightarrow \quad \exists\, \xi \in \mathbb{R}^d: \quad \pi^C + \xi Y \leq \dfrac{C}{1+r}$ P-a.s.

$\mathcal{M}_{\pi^C}$ is AF $\Rightarrow \quad \pi^C + \xi Y = \dfrac{C}{1+r}$ P-a.s.

$\Rightarrow$ C is replicable $\quad \square$(a)

(b) By (a) we only have to show that in the none-singleton case the set $\Pi(C)$ is an open interval.

$\mathcal{P}$ is convex. $\Rightarrow$ $\mathcal{P} \cap \{P^* \approx P \mid E^*[\frac{C}{1+r}] < \infty\}$

is convex.

$\underset{\uparrow}{\Rightarrow}$ $\Pi(C)$ is an interval.

Prop. 24. To show $\Pi_{inf}(C), \Pi_{sup}(C) \notin \Pi(C)$.

Prop. 25 $\Rightarrow$ $\exists\, \xi \in \mathbb{R}^d : \Pi_{inf}(C) + \xi Y \leq \frac{C}{1+r}$ P-as.

No "=" P-as, because C is not replicable

$\Rightarrow$ $\mathcal{M}\mathcal{M}_{\Pi_{inf}(C)}$ has AO. $\Rightarrow$ $\Pi_{inf}(C) \notin \Pi(C)$

Analogously $\Pi_{sup}(C) \notin \Pi(C)$ $\square$

Lecture 4, 24.02.2022

Examples 29: Let $(\underline{\Pi}, \underline{S}, P)$ be arbitrage free.

We study $\Pi(C)$ for (a) $C = C^{call}$ and (b) $C = C^{put}$.

(a) $C = C^{call} = (S^{(i)} - K)^+$ with strike $K > 0$, $i > 0$,

and assume r is constant.

Then we have a for $P^* \in \mathcal{P}$:

~~$\frac{(\Pi^{(i)} - K)^+}{1+r} \underset{\uparrow}{\leq} \frac{1}{1+r} E^*[$~~

~~Jensen's inequality~~ ($x \mapsto (x-K)^+$ is convex)

$$\left(\pi^{(1)} - \frac{K}{1+r}\right)^{+} \underset{\uparrow}{\leq} E^*\left[\left(\frac{S^{(1)}}{1+r} - \frac{K}{1+r}\right)^{+}\right]$$

Jensen's inequality, because $x \mapsto \left(x - \frac{K}{1+r}\right)^{+}$ is convex.

$$= E^*\left[\frac{C^{call}}{1+r}\right] \underset{\uparrow}{\leq} E^*\left[\frac{S^{(1)}}{1+r}\right] = \pi^{(1)}$$

$C^{call} \leq S^{(1)}$

$$\Rightarrow \left(\pi^{(1)} - \frac{K}{1+r}\right)^{+} \leq \pi_{\inf}(C) \leq \pi_{\sup}(C) \leq \pi^{(1)}.$$

(b) By the put-call-parity we have:

$$\left(\frac{K}{1+r} - \pi^{(1)}\right) \leq \pi_{\inf}(C^{put}) \leq \pi_{\sup}(C^{put}) \leq \frac{K}{1+r}.$$

<u>Proof:</u> Let $l := \sup\{a \in \mathbb{N} \mid \exists A_1, \dots, A_a \in \mathcal{A}:$

$A_1 \dot\cup \dots \dot\cup A_a = X$ and $Q(A_j) > 0$ for all $j\}$

If $l > m$ then

$\dim_{\mathbb{R}} L^0(X, \mathcal{A}, P) \geq l > m$ ↯.

$\Rightarrow \quad l \leq m$.

Take $A_1 \dot\cup \dots \dot\cup A_l = X$, $Q(A_j) > 0 \; \forall j$.

Then every A_j is an atom, otherwise

if $A_j = A_{j,1} \dot\cup A_{j,2}$ with $Q(A_{j,1}) > 0$

$Q(A_{j,2}) > 0$

then we get a contradiction to the maximality of l.

<u>Exercise:</u> $\forall f \in \mathcal{L}^0(X, \mathcal{A}, Q)$:

$\exists y_1, \dots, y_l: \quad f = \sum_{j=1}^{l} y_j \mathbb{1}_{A_j} \quad Q\text{-a.s.}$

$\Rightarrow \dim_{\mathbb{R}} L^0(X, \mathcal{O}_1, a) = l$

$\Rightarrow l = m$ $\square$

Thus in a complete market model we do not have many different events.

Let $\mathcal{F}$ be equal to $\sigma(S^{(0)}, \dots, S^{(d)})$. !

33 **Prop 1.40:** An AF MM is complete

$\Leftrightarrow |\mathcal{P}| = 1.$

Proof: "$\Rightarrow$" Take $A \in \mathcal{F} = \sigma(S^{(0)}, \dots, S^{(d)})$.

$C := \mathbb{1}_A \cdot S^{(0)}_1$ replicable, so has a unique AF price π^C:

$$\Rightarrow \{\pi^C\} = \Pi(C) = \left\{ E^*\left[\frac{C}{1+r}\right] \,\middle|\, P^* \in \mathcal{P} \text{ and } E^*\left[\frac{C}{1+r}\right] < \infty \right\}$$

Note $E^*\left[\frac{C}{1+r}\right] < \infty$ is fullfilled, because

$\left|\frac{C}{1+r}\right| \leq 1$ ~~a.s.~~ $\cancel{\text{ }x \neq 0}$.

$$\Rightarrow \{\pi^C\} = \Pi(C) = \left\{ E^*\left[\frac{C}{1+r}\right] \,\middle|\, P^* \in \mathcal{P} \right\}$$

$$= \{ E^*[\mathbb{1}_A] \mid P^* \in \mathcal{P} \}$$

$$= \{ P^*(A) \mid P^* \in \mathcal{P} \}$$

$\Rightarrow |\mathbb{P}| = 1$

"$\Leftarrow$" Take a contingent claim C.

$\Rightarrow \quad \Pi(C) = \left\{ E^{\ast}\left[\frac{C}{1+r}\right] \,\middle|\, P^{\ast} \in \mathbb{P} \,,\, E^{\ast}\left[\frac{C}{1+r}\right] < \infty \right\}$

This set is not empty.

$= \left\{ E^{\ast}_{Q}\left[\frac{C}{1+r}\right] \right\}$, where $\mathbb{P} = \{Q\}$.

$\Rightarrow \quad C$ is replicable by Prop. 28 (a).

$\square$

I.5. Return & Leverage effect

Def. 34: Given an A.F. MM and $V \in \mathcal{V}_{\text{r.t.}\frac{\pi(V)}{\pi}}$.

We call $R(V) := \frac{V - \pi(V)}{\pi(V)}$ the return of V. (Note the price of V is unique!)

Example 35: (a) $R(S^0) = r$

(b) If $V = \sum_{k=1}^{l} \alpha_k V_k$, $\alpha_k \in \mathbb{R}$, $V_k \in \mathcal{V}$, $\pi(V_k) \neq 0$.

Then $R(V) = \sum_{k=1}^{l} \beta_k R(V_k)$

with $\beta_k = \frac{\alpha_k \pi(V_k)}{\sum_{j=1}^{l} \alpha_j \pi(V_j)}$,

under the assumption that

~~$P(V \neq 0) \neq 0$.~~ $\pi(V) \neq 0$.

(c) $V_i := S^{(i)}$, $i = 0, \dots, d$. $\underline{\xi} \in \mathbb{R}^{d+1}$, $\pi^{(i)} \neq 0$,

$V := \underline{\xi}\,\underline{S} = \sum_{i=0}^{d} \xi^{(i)} V_i$.

Assume $\underline{\xi}\,\underline{\pi} \neq 0$. Then we have

$$R(V) = \sum_{i=0}^{d} \frac{\xi^{(i)} \cdot \pi^{(i)}}{\underline{\xi}\,\underline{\pi}} R(S^{(i)}).$$

End of Lecture 5. 01.03.2022

Proposition 36: Let $(\pi, \underline{S}, P)$ be A F and r be deterministic and $V \in \mathcal{V}$ with $\pi(V) \neq 0$.
Then $E^*[R(V)] = r$, if $\pi^{(i)} \neq 0$ for all i.

Proof: $V = \underline{\xi}\,\underline{S}$

$$\Rightarrow E^*[R(V)] = \sum_{i=0}^{d} \frac{\xi^{(i)}\pi^{(i)}}{\pi(V)} E^*[R(S^{(i)})]$$

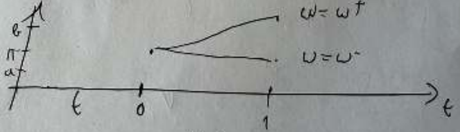
$$= \sum_{i=0}^{d} \frac{\xi^{(i)}\pi^{(i)}}{\pi(V)} r = r. \quad \square$$

Example 37: (a) $\Omega = \{\omega^+, \omega^-\}$ $P(\{\omega^+\}) = p > 0$
and $P(\{\omega^-\}) = 1-p > 0$

r deterministic.

$$S^{(1)}(\omega) = \begin{cases} b, & \omega = \omega^+ \\ a, & \omega = \omega^- \end{cases} \qquad a < b$$

price $\pi = \pi^{(1)} \in \mathbb{R}_+$



(compare with Ex. 16.)
We need an A.F. MM.
So $\pi \in \left] \frac{a}{1+r}, \frac{b}{1+r} \right[$.

and we have a unique martingale measure P^*

$p^* = P^*(\{\omega^+\}) \in]0,1[$

$\mathcal{P} = \{P^*\} \Rightarrow (\Pi, \mathcal{J}, P)$ is complete

Let C be a contingent claim

Completeness $\Rightarrow$ C is replicable $\Rightarrow \exists\, \xi^{(0)}, \xi \in \mathbb{R}$

$$C = \xi^{(0)}(1+r) + \xi^{(1)} S^{(1)} \qquad (*)$$

$$\xrightarrow[\substack{1^{st}\ \omega = \omega^+ \\ 2^{nd}\ \omega = \omega^-}]{} \quad \frac{C(\omega^+) - C(\omega^-)}{b-a} = \xi$$

and $\xi^{(0)}$ is given by the equation $(*)$.

$$\xi^{(0)} = \frac{C(\omega^+) - \frac{C(\omega^+) - C(\omega^-)}{b-a}\, b}{1+r}$$

$$= \frac{b\,C(\omega^-) - a\,C(\omega^+)}{(1+r)(b-a)}$$

$$\Rightarrow \quad \pi(C) = \xi^{(0)} + \xi\,\pi$$

$$= \frac{1}{b-a}\left(\frac{b\,C(\omega^-) - a\,C(\omega^+)}{1+r} + \pi\left(C(\omega^+) - C(\omega^-)\right)\right)$$

$$= \frac{1}{(b-a)(1+r)}\left(C(\omega^+)\left((1+r)\pi - a\right) - C(\omega^-)\left((1+r)\pi - b\right)\right)$$

$$\underset{\substack{\uparrow \\ \pi(r+1) = p^* b + (1-p^*)a}}{=} \frac{1}{1+r}\left(C(\omega^+)\,p^* + (1-p^*)\,C(\omega^-)\right).$$

a) Let C be a call option with strike $K \in [a, b]$

$$C^{call} = (S - K)^+$$

$$\Pi(C^{call}) = C^{call}(\omega^+) \frac{((1+r)\Pi - a)}{(b-a)(1+r)}$$

$$= \frac{(b-K)}{(b-a)} \Pi - \frac{(b-K)a}{(b-a)(1+r)}$$

$\Pi(C^{call})$ does not depend on p.

Naiv: $E\left[\frac{C^{call}}{1+r}\right] = \frac{p(b-K)}{(1+r)}$

depends on p.

Further the price increases if r increases and the naiv price decreases if r increases.

) Now we specify further:

$r = 0$, $\Pi = 100$, $b = 110$, $a = 90$.

$$R(S)(\omega^+) = \frac{110 - 100}{100} = 10\%.$$

$$R(S)(\omega^-) = \frac{90 - 100}{100} = -10\%.$$

$K = 100$.

$$R(C^{call})(\omega^+) = \frac{C^{call}(\omega^+) - \Pi(C^{call})}{\Pi(C^{call})}$$

$$\pi(C^{call}) = \frac{1}{2}\cdot 100 - \frac{1}{2}\,90$$
$$= 5$$

$$R(C^{call})(\omega^+) = \frac{10-5}{5} = 100\%$$
$$R(C^{call}_C)(\omega^-) = \frac{0-5}{5} = -100\%.$$

This is the "leverage effect".

(d) Example with a put option

Our portfolio: $1 \quad S$
$1 \quad C^{put}$.

$$\tilde{C} := C^{put} + S = (K-S)^+ + S$$

$$R(\tilde{C}) = \frac{\pi(C^{put})}{\pi(\tilde{C})} R(C^{put}) + \frac{\pi(S)}{\pi(\tilde{C})} R(S)$$

C	$\pi(\ )$
S	$\pi = 100$
C^{call}	5
C^{put}	5
$\tilde{C}$	105

$(\pi(C^{put}) - \pi(C^{call}) = \pi(K-S) = 100 - 100 = 0.)$

$$R(\tilde{C})(\omega) = \begin{cases} \frac{5}{105}(-1) + \frac{100}{105}\frac{10}{100} = \frac{1}{21} \approx 4{,}76\%, & \omega = \omega^+ \\ \frac{5}{105}\cdot 1 + \frac{100}{105}\cdot\frac{(-10)}{100} = -\frac{1}{21} \approx -4{,}76\%, & \omega = \omega^- \end{cases}$$

We get a smaller risk compared of just holding S.

I.6 Random start.

Setting 38:
Now we consider instead of $\mathbb{R}$ a random start, because we want to analyse a multi period model. (Think that $t=0$ and $t=1$ are two times in the future.) $(\Omega, \mathcal{F}, P)$ probability space.

$t=0$: $\mathcal{F}_0$, $\underline{S}_0 = (S_0^{(0)}, S_0^{(1)}, S_0^{(2)}, \dots, S_0^{(d)}) : \Omega \to \mathbb{R}_+^{d+1}$ $\mathcal{F}_0$-measurable.

$t=1$: $\mathcal{F}$, $\underline{S}_1 = (S_1^{(0)}, S_1^{(1)}, S_1^{(2)}, \dots, S_1^{(d)}) : \Omega \to \mathbb{R}_+^{d+1}$ $\mathcal{F}$-measurable.

Condition:
$P(S_0^{(0)} > 0, S_1^{(0)} > 0) = 1.$

discounted payoff of ith ~~security~~ asset $X_t^{(i)} := \frac{S_t^{(i)}}{S_t^{(0)}}$

$$Y^{(i)} := X_1^{(i)} - X_0^{(i)} = \frac{S_1^{(i)} - \frac{S_1^{(0)}}{S_0^{(0)}} S_0^{(i)}}{S_1^{(0)}}$$

discounted net gain of the ith asset.

The portfolio $\underline{\xi} = (\xi^{(0)}, \xi^{(1)}, \dots, \xi^{(d)})$ is $\mathcal{F}_0$-measurable.

Def. 39: A portfolio $\underline{\xi}$ is called AO. if

$\underline{\xi}\,\underline{S}_0 \le 0$ P-as.

$P(\underline{\xi}\,\underline{S}_1 > 0) > 0$

$\underline{\xi}\,\underline{S}_1 \ge 0$ P-as.

end of Lecture 6. 03.03.2022

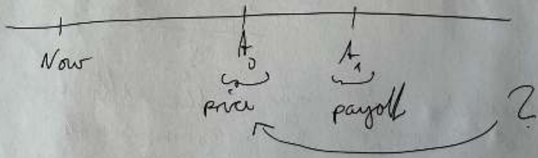
Question 40: What should a martingale measure be here?

Interlude: Conditional expectation

Motivation 41: (Let r be 0.)

The price of an asset today should be the expectation of the payoff wrt. a martingale measure.

What about timepoints in the future? What is the price of an asset in a time in the future wrt. its payoff in the further future?



The price at t_0 is random, because it lies in the future.

Knowledge:	today	t_0	t_1
	$\{\emptyset, \Omega\}$	$\mathcal{F}_0$	$\mathcal{F}$.

We know ~~least~~ more at A_0 than today $\mathcal{F}_0 \supseteq \{\emptyset, \Omega\}$

—— " —— less —" ——— at A_1. $\mathcal{F}_0 \subseteq \mathcal{F}$.

We want " $X_{A_0}(\omega_0) = E^*\left[X_{A_1} \mathbb{1}\{X_{A_0} = X_{A_1}(\omega_0)\}\right]$

$$\cdot \frac{1}{P^*(\{\omega \in \Omega \mid X_{A_1}(\omega) = X_{A_0}(\omega_0)\})}$$

This is the right choice if $P^*(\{\omega \in \Omega \mid X_{A_0}(\omega) = X_{A_0}(\omega_0)\})$ is ~~less~~ positive. ~~measure~~.

But if $P^{X_{A_0}} \ll \lambda\big|_{\mathcal{B}(\mathbb{R}^{d+1})}$, then those ~~proba~~ probabilities are zero!

Thus we need:

$$E^*\left[X_{A_1} \mathbb{1}_{\{X_{A_0} \in]x, x'[\}}\right] \frac{1}{P^*(\{\omega \in \Omega \mid X_{A_0} \in]x, x'[\})}$$

$$= E^*\left[X_{A_1} \mathbb{1}_{\{X_{A_0} \in]x, x'[\}}\right] \frac{1}{P^*(\{\omega \in \Omega \mid X_{A_0} \in]x, x'[\})}$$

~~We consider a measurable space $(\Omega, \mathcal{F})$, $\mathcal{F}_0 \subseteq \mathcal{F}$, and a measure~~

Def 42: 1) Let X be $\mathcal{F}$ measurable with $E[|X|] < \infty$. A ~~random~~ $\mathcal{F}_0$ - measurable $\mathcal{L}^1$-random variable $X_0 : \Omega \to \mathbb{R}$ is called the "conditional expectation of X wrt. $\mathcal{F}_0$" if X_0 satisfies:

$$E[X_0 \mathbb{1}_A] = E[X \mathbb{1}_A] \ \forall \ A \in \mathcal{F}_0.$$

2) Similar definition for $X \geq 0$.

Prop 43: Let X be $\mathcal{F}$- measurable with $E[|X|] < \infty$. Then $\exists$ a conditional expectation X_0 of X w.r.t. $\mathcal{F}_0$, and

a) X_0 is unique P-as-equivalence.

b) $E[X_0] = E[X]$ and $E[|X|] \geq E[|X_0|]$.

Proof: We prove a) first.

a) X_0, Y_0 two cond. exp. of X wrt. $\mathcal{F}_0$.

$$E[(X_0 - Y_0) \mathbb{1}_{\underbrace{\{X_0 > Y_0\}}_{A}}] =$$

$$= E[X_0 \mathbb{1}_A] - E[Y_0 \mathbb{1}_A] = E[X \mathbb{1}_A] - E[X \mathbb{1}_A]$$

$= 0.$

$\Rightarrow (X_0 - Y_0) \mathbb{1}_{\{X_0 > Y_0\}} = 0$ P-as.

$\square$

We now prove the existence.

By $X = X^+ - X^-$, we only need to consider $X \geq 0$. We define

$$\nu(A) := E[X \mathbb{1}_A].$$

ν is a ~~xxxxx~~ measure on $\mathcal{F}_0$, $\nu \ll \mathbb{P}|_{\mathcal{F}_0}$

Radon - Nikodym $\Rightarrow \exists f : \Omega \longrightarrow \mathbb{R}_+$: $\mathcal{F}_0$-measurable: $\forall_{A \in \mathcal{F}_0}$: $\nu(A) = E[f \cdot \mathbb{1}_A]$.

$X_0 := f$ is a conditional expectation of X w.r.t. $\mathcal{F}_0$.

b) $E[X_0] = E[X_0 \cdot \mathbb{1}_\Omega] = E[X \mathbb{1}_\Omega] = E[X]$.

and let X_0^+ be a cond. exp. of X^+ w.r.t. $\mathcal{F}_0$

X_0^- X^-

Then $X_0^+ - X_0^-$ is a cond. exp. of X w.r.t $\mathcal{F}_0$.

So we have

$$E[|X_0|] \leq E[X_0^+ + X_0^-] = E[X_0^+] + E[X_0^-]$$
$$= E[X^+] + E[X^-] = E[|X|]. \square$$

end of Lecture 7; 8.3.22

Notation 4.4: We write $E[X \mid \mathcal{F}_0]$ for the conditional exp. of X wrt. $\mathcal{F}_0$ in $L^1(\mathcal{F}_0, P)$.
(or $L^0(\mathcal{F}_0, P)$ in the $X \geq 0$ ($X \leq 0$) case)

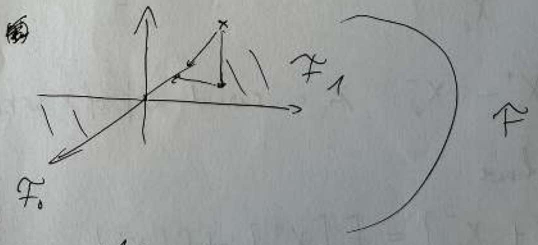
We see $E[X \mid \mathcal{F}_0]$ as both ~~the~~ a random variable as well as an equivalence class.

Properties of conditional expectations 4.5: Take $X \in \mathcal{L}^1(\mathcal{F}, P)$.

(1) Suppose X is $\mathcal{F}_0$-measurable.
Then $X = E[X \mid \mathcal{F}_0]$ P-a.s.

(2) transitivity property: $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}$

$$E[X \mid \mathcal{F}_0] = E[E[X \mid \mathcal{F}_1] \mid \mathcal{F}_0]$$



$$L^1(\mathcal{F}, P) \supseteq L^1(\mathcal{F}_1, P) \supseteq L^1(\mathcal{F}_0, P)$$

(3) Let $Y_0 \in \mathcal{L}^0(\mathcal{F}_0, P)$ and $XY_0 \in \mathcal{L}^1(\mathcal{F}, P)$.

Then $E[X|\mathcal{F}_0]Y_0 \in \mathcal{L}^1(\mathcal{F}_0, P)$ and

$E[XY_0|\mathcal{F}_0] = E[X|\mathcal{F}_0]Y_0$ P-a.s.

(4) <u>Linearity:</u> $X, Y \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$, $\alpha, \beta \in \mathcal{L}^\infty(\Omega, \mathcal{F}_0, P)$

Then $\alpha E[X|\mathcal{F}_0] + \beta E[Y|\mathcal{F}_0]$

$= E[\alpha X + \beta Y | \mathcal{F}_0]$ P-a.s.

(5) <u>orthogonal projection:</u>

Consider $X \in L^2(\mathcal{F}, P)$.

On $L^2(\mathcal{F}, P)$ we have the scalar product $\langle X, Y \rangle := E[XY]$.

(This is not a scalar product on $\mathcal{L}^2(\mathcal{F}, P)$! Why?)

<u>Claim:</u> On $\mathcal{F}_0 \subseteq \mathcal{F}$ we have:

$E[X|\mathcal{F}_0]$ is the orthogonal projection of X onto $L^2(\mathcal{F}_0, P)$ w.r.t. $\langle \cdot, \cdot \rangle$, i.e.

we have

$$< E[X|\mathcal{F}_0] - X, Y_0 > = 0 \quad \forall_{Y_0 \in L^2(\mathcal{F}_0, P)}.$$

(Note that an orthogonal projection of X onto $L^2(\mathcal{F}_0, P)$ is unique, because

two orthogonal projections $\tilde{Y}_0, \tilde{X}_0$ satisfy

$$< \tilde{X}_0, Y_0 > = < \tilde{Y}_0, Y_0 > \quad \forall_{Y_0 \in L^2(\mathcal{F}_0, P)}, \text{ so}$$

$$\underbrace{< \tilde{X}_0 - \tilde{Y}_0, \tilde{X}_0 - \tilde{Y}_0 >}_{= \| \tilde{X}_0 - \tilde{Y}_0 \|_2} = 0 \Rightarrow \tilde{X}_0 = \tilde{Y}_0 \text{ in } L^2(\mathcal{F}_0, P))$$

<u>Proof</u>: (1) If X is already $\mathcal{F}_0$-measurable then we just can take $X_0 = X$ for a cond. exp. of X wrt. $\mathcal{F}_0$, because

$X \in \mathcal{L}^1(\mathcal{F}_0, P)$ and

$E[X \mathbb{1}_A] = E[X \mathbb{1}_A] \; \forall A \in \mathcal{F}_0.$

(2) X, $E[X|\mathcal{F}_1]$ and $E[X|\mathcal{F}_0]$ are $\mathcal{L}^1$ because the two latter are cond. exp. of an $\mathcal{L}^1$-

variable

For $A \in \mathcal{F}_0$ we have

$$E[E[X|\mathcal{F}_1]\mathbb{1}_A] \overset{A\in\mathcal{F}_1}{=} E[X\mathbb{1}_A] \overset{A\in\mathcal{F}_0}{=} E[E[X|\mathcal{F}_0]\mathbb{1}_A]$$

Further $E[X|\mathcal{F}_0]$ is $\mathcal{F}_0$-measurable.

$$\Longrightarrow \quad E[E[X|\mathcal{F}_1]|\mathcal{F}_0] = E[X|\mathcal{F}_0] \text{ P-a.s.}$$

3) Write X_0 for $E[X|\mathcal{F}_0]$.

(3a) Take at first $Y_0 = \mathbb{1}_B$, $B \in \mathcal{F}_0$.

(i) Then $|X_0\mathbb{1}_B| \leq |X_0|$ and $X_0\mathbb{1}_B$ is $\mathcal{F}_0$-measurable, so $X_0\mathbb{1}_B \in \mathcal{L}^1(\mathcal{F}_0, P)$

(ii) $A \in \mathcal{F}_0$: $\quad E[(X_0\mathbb{1}_B)\mathbb{1}_A] = E[X_0\mathbb{1}_{\underbrace{A\cap B}_{\in\mathcal{F}_0}}]$

$$\underset{X_0 = E[X|\mathcal{F}_0]}{=} E[X\mathbb{1}_{A\cap B}] = E[(X\mathbb{1}_B)\mathbb{1}_A]$$

(i)$\wedge$(ii) $\Longrightarrow \quad E[X|\mathcal{F}_0]Y_0 = E[XY_0|\mathcal{F}_0]$ P-a.s.

(3b) We have $X = X^+ - X^-$ and $Y_0 = Y_0^+ - Y_0^-$

and $E[X^+|\mathcal{F}_0] = X_0^+$ P-a.s. and

$$E[X^-|\mathcal{F}_0] = X_0^- \text{ P-a.s.}$$

We have to show $E[X^{\varepsilon} Y_0^{\delta}] = X_0^{\varepsilon} Y_0^{\delta}$ P-as.
for all $\varepsilon, \delta \in \{+,-\}$.

We only consider $\varepsilon = \delta = +$, i.e. the case
$X \geq 0$ and $Y_0 \geq 0$.

Y_0 is of the for $\sum_{j=1}^{\infty} \alpha_j 1\!\!1_{B_j}$ in $L^1(\mathcal{F}_0, P)$
with $\alpha_j \geq 0$ and $B_j \in \mathcal{F}_0$.

(3b i) By monoton convergence we have

$$E[X_0 Y_0] = E\Big[X_0 \sum_{j=1}^{\infty} \alpha_j 1\!\!1_{B_j}\Big]$$

monoton convergence

$$\overset{\downarrow}{=} \sum_{j=1}^{\infty} \alpha_j E[X_0 1\!\!1_{B_j}] = \sum_{j=1}^{\infty} \alpha_j E[X 1\!\!1_{B_j}]$$

$$\overset{\downarrow}{=} E\Big[\sum_{j=1}^{\infty} X \alpha_j 1\!\!1_{B_j}\Big] = E[X Y_0] \underset{\substack{\uparrow \\ X Y_0 \in \mathcal{L}^1(\mathcal{F}, P)}}{<} \infty$$

· $X_0 Y_0$ is $\mathcal{F}_0$- measurable.

(3b ii) $A \in \mathcal{F}_0$: $E[X_0 Y_0 1\!\!1_A] \underset{\substack{\uparrow \\ \text{(3b i) with} \\ Y_0 1\!\!1_A \text{ instead of } Y_0}}{=} E[X Y_0 1\!\!1_A]$

(3b i) and (3b ii) $\Rightarrow$ $X_0 Y_0 = E[XY_0 \mid \mathcal{F}_0]$ P-a.s.

(4) By (3) we only need to show additivity.
$X, Y \in \mathcal{L}^1(\mathcal{F}, P)$.

(4i) · $E[X \mid \mathcal{F}_0] + E[Y \mid \mathcal{F}_0]$ is $\mathcal{F}_0$-measurable, because the summands are.

· $$\left| E[X \mid \mathcal{F}_0] + E[Y \mid \mathcal{F}_0] \right| \leq \left| E[X \mid \mathcal{F}_0] \right| + \left| E[Y \mid \mathcal{F}_0] \right|$$

and the latter has finite expectation.

$$\Rightarrow E[X \mid \mathcal{F}_0] + E[Y \mid \mathcal{F}_0] \in \mathcal{L}^1(\mathcal{F}_0, P).$$

(4ii) Take $A \in \mathcal{F}_0$.

$$E\left[\left(E[X \mid \mathcal{F}_0] + E[Y \mid \mathcal{F}_0]\right) \mathbb{1}_A\right]$$
$$= E\left[E[X \mid \mathcal{F}_0] \mathbb{1}_A\right] + E\left[E[Y \mid \mathcal{F}_0] \mathbb{1}_A\right]$$
$$= E[X \mathbb{1}_A] + E[Y \mathbb{1}_A] = E[(X+Y) \mathbb{1}_A].$$

(4i) + (4ii) $\Rightarrow$ $E[X+Y \mid \mathcal{F}_0] = E[X \mid \mathcal{F}_0] + E[Y \mid \mathcal{F}_0]$.

(5) The main part is to show that $X_0 := E[X \mid \mathcal{F}_0] \in \mathcal{L}^2(\mathcal{F}_0, P)$, i.e. square integrable.

Once we now this then we have for $Y_0 \in \mathcal{L}^2(\mathcal{F}_0, P)$

$$\langle X - X_0, Y_0 \rangle = \langle X - X_0, Y_0 \rangle$$

$$= \langle X, Y_0 \rangle - \langle X_0, Y_0 \rangle \underset{\substack{\uparrow \\ (4) \text{ and } 43(e)}}{=} \langle X_0, Y_0 \rangle - \langle X_0, Y_0 \rangle$$

$$= 0$$

We only need to consider the case $X \geq 0$.

We then have $X_0 \geq 0$ P-as.

Take a sequence $X^{(n)}$ of elements of $\mathcal{L}^\infty(\mathcal{F}, P)$ s.t. $0 \leq X^{(n)} \nearrow X$.

Let $X_0^{(n)} = E[X^{(n)} \mid \mathcal{F}_0]$ and $X_0 = E[X \mid \mathcal{F}_0]$.

Exercise $X_0^{(n)} \leq X_0^{(n+1)} \leq X_0$ and $X_0^{(n)} \in \mathcal{L}^\infty(\mathcal{F}_0, P)$

$$0 \leq E[(X^{(n)} - X_0^{(n)})^2] = E[(X^{(n)})^2 - 2X^{(n)}X_0^{(n)} + (X_0^{(n)})^2]$$

$$= E[(X^{(n)})^2] + E[(X_0^{(n)})^2] - 2E[X^{(n)}X_0^{(n)}]$$

$$= E[(X^{(n)})^2] + E[(X_0^{(n)})^2] - 2E[X_0^{(n)} X_0^{(n)}]$$

$\uparrow$

$X_0^{(n)}$ is ~~the~~ a cond. exp. of $X^{(n)}$ w.r.t. $\mathcal{F}_0$ and (4)

$$= E[(X^{(n)})^2] - E[(X_0^{(n)})^2]$$

$$\Rightarrow \quad E[(X_0^{(n)})^2] \leq E[(X^{(n)})^2]$$

$n \to \infty$ and monotone convergence

$$\Rightarrow \quad E[X_0^2] \leq E[X^2]. \qquad \square$$

End of Lecture 8 10.3.22

Remark 46: a) 45 (4) can be stated in the following way: We know that $L^2(\mathcal{F}, P)$ is a Hilbert space, i.e.

- it has a scalar product $\langle , \rangle_2$

$$\langle X, Y \rangle_2 := E[XY]$$

- it has a norm given by the scalar product $\|X\|_2 := \sqrt{\langle X, X \rangle_2}$
- $L^2(\mathcal{F}, P)$ with $\|X\|_2$ is complete. (Every Cauchy sequence converges.)

Now given $X \in L^2(\mathcal{F}, P)$, then $X_0 := E[X \mid \mathcal{F}_0]$ is the unique element of $L^2(\mathcal{F}_0, P)$, which minimizes the distance to X in $L^2(\mathcal{F}_0, P)$.

<u>Proof</u>: For $Y_0 \in L^2(\mathcal{F}_0, P)$ we have

$$\|X - Y_0\|_2^2 \underset{\substack{\uparrow \\ X - X_0 \perp X_0 - Y_0 \\ \text{by } 45(4)}}{=} \langle X - X_0, X - X_0 \rangle_2 + \langle X_0 - Y_0, X_0 - Y_0 \rangle_2$$

$$\geq \|X - X_0\|_2^2$$

and if Y_0 minimizes $\|X - Z_0\|_2$ in $L^2(\mathcal{F}_0, P)$, then $\langle X_0 - Y_0, X_0 - Y_0 \rangle_2 = 0$ and thus $X_0 = Y_0$ in $L^2(\mathcal{F}_0, P)$. $\square$

<u>Remark 47</u>: (a) Suppose $\mathcal{F}_0$ is given by P-atoms $(A_n)_{n \in \mathbb{N}}$

$\mathcal{F}_0 = \{ B \cup N \mid N \in \mathcal{F}_0 \text{ zero-set},$

B is a union of A_i's $\}$.

$$E[X \mid \mathcal{F}_0](\omega) = E[X ; A_i] \text{ if } \omega \in A_i$$
$$:= E[X \mathbb{1}_{A_i}] \cdot \frac{1}{P(A_i)}$$

P-a.s.

(exercise)

(b) change of measure ~~A bit more general than tos~~. Suppose we have $P \ll Q$ on $\mathcal{F}$, i.e. $\exists$ Radon-Nikodym density $\frac{dP}{dQ}$.

Then $P|_{\mathcal{F}_0} \ll Q|_{\mathcal{F}_0}$ and $\frac{d(P|_{\mathcal{F}_0})}{d(Q|_{\mathcal{F}_0})} = E_Q\left[\frac{dP}{dQ} \middle| \mathcal{F}_0\right]$

(exercise)

and we get for $X \in L^1(\mathcal{F}, P)$

$$E_P[X \mid \mathcal{F}_0] = \frac{1}{E_Q\left[\frac{dP}{dQ} \mid \mathcal{F}_0\right]} E_Q\left[X \frac{dP}{dQ} \middle| \mathcal{F}_0\right]$$

(exercise)

finish of interlude.

Remark 48: We only need to search for bounded portfolios if we look for an AO, i.e. if there is an AO then there is an AO $\underline{\xi}$ with $\xi^{(i)} \in L^\infty(\Omega, \mathcal{F}_0, P)$ for all $i = 0, \dots, d$.

(Why? Exercise)

We want to prove the FTAP for random starts.

At first we do not need to consider the numéraire $S_t^{(0)}$: $\quad X_t^{(i)} := \dfrac{S_t^{(i)}}{S_t^{(0)}}$, $\quad Y^{(i)} = X_1^{(i)} - X_0^{(i)}$

discounted net gain.

$$\mathcal{K} := \left\{ \xi Y \;\middle|\; \xi \in L^0(\mathcal{F}_0, P, \mathbb{R}^d) \right\}$$

The market model is AF iff $\mathcal{K} \cap L^0_+(\mathcal{F}_1, P) = \{0\}$

Def 49: A probability measure Q on $\mathcal{F}$ is called "martingale measure" for the given market model $(\underline{S}_{\mathcal{T}}, P, \mathcal{F}_0 \subseteq \mathcal{F})$ if $Q \approx P$ and

(a) $E_Q[X_t^{(i)}] < \infty$, for $i = 1, \dots, d$ and $t \in \{0, 1\}$, and

(b) $E_Q[X_1^{(i)} \mid \mathcal{F}_0] = X_0^{(i)}$, for $i = 1, \dots, d$

$\mathcal{P} :=$ set of martingale measures for $(\underline{S}_{\mathcal{T}}, P, \mathcal{F}_0 \subseteq \mathcal{F})$

Proposition 50 (1.46) The following assertions are equivalent.

1° $\mathcal{R} \cap L_+^0(\mathcal{F}_1, P) = \{0\}$ (AF)

2° $(\mathcal{R} - L_+^0) \cap L_+^0 = \{0\}$

3° $\exists\, P^* \in \mathcal{P}$ with bounded density $\frac{dP^*}{dP}$

4° $\mathcal{P} \neq \emptyset$.

Remark 51: If $\mathcal{F}_0$ is generated by finitely many atoms then Prop 50 follows from Prop 15.

Proof (of Prop 50)

$4^\circ \Rightarrow 1^\circ$ Assume the market model is not AF.

Then, by Remark 48, there is a bounded AO $\xi \in L^0(\mathcal{F}_0, P, \mathbb{R}^{d+1})$.

Then $\xi Y \geq 0$ P-a.s. and $P(\xi Y > 0) > 0$.

(Why? Repeat the exercise!)

Take $Q \in \mathbb{P}$

We have $\xi X_0 = E_Q[\xi X_1 | \mathcal{F}_0]$

and therefor $\underbrace{E[\xi Y] = 0}$

$\xi Y \geq 0$ P-a.s. $\Rightarrow$ $\xi Y = 0$ P-a.s. $\lightning$

$1^\circ \Leftrightarrow 2^\circ$ ✓

$3^\circ \Rightarrow 4^\circ$ ✓

We are left to prove $(2^\circ \Rightarrow 3^\circ)$, but this is more difficult. We give the proof step by step.

Step 1: By changing the measure P using a bounded density we can acheive $|\underline{X}_0|_\infty + |\underline{X}_1| \in L^1(\mathcal{F}, P)$.

$(|\cdot|_\infty$ is the maximum norm in $\mathbb{R}^d)$

Step 2: $\mathcal{C} := (\mathcal{R} - L^0_+) \cap L^1$ is a convex cone.

Convex: $0 \le \lambda \le 1$ real number,

$$\zeta Y - \alpha,\ \xi Y - \beta \in \mathcal{C} \text{ with } \alpha, \beta \in L^0_+$$

$$\Rightarrow L^1 \ni \lambda(\zeta Y - \alpha) + (1-\lambda)(\xi Y - \beta)$$

$$= \underbrace{(\lambda \zeta + (1-\lambda)\xi) Y}_{\in \mathcal{R}} - \underbrace{(\lambda \alpha + (1-\lambda)\beta)}_{\in L^0_+}$$

$\in \mathcal{C}$.

Cone property: $0 \leq \lambda$ real number

$\xi Y - \alpha \in \mathcal{C}$, $\alpha \in L^0_+$

$$\Rightarrow \underbrace{\lambda \xi Y}_{\tilde{\xi}} - \underbrace{\lambda \alpha}_{\in L^0_+} \in L^1$$

is still in $\mathcal{C}$.

end of Lecture 9. 15.03.2022.

Step 3: We can use $\mathcal{C}$ to look for martingale measures for $\underline{X}_*$.

Lemma 52: Let $c \in \mathbb{R}_+$ and $z \in L^\infty(\mathcal{F}, P)$ such that $E[zW] \leq c \quad \forall W \in \mathcal{C}$.

Then (a) $E[zW] \leq 0 \quad \forall W \in \mathcal{C}$

(b) $z \geq 0$ P-a.s.

(c) If $z \not\equiv 0$ then $\frac{dQ}{dP} = \frac{z}{E[z]}$ defines a martingale measure for $\underline{X}_*$

(d) If $P(z \neq 0) = 1$ then $\frac{dQ}{dQ} = \frac{z}{E[z]}$ defines a martingale measure for the market model, i.e. $Q \in \mathbb{P}$.

Proof: a) trivial

b) $-1\!\!1_{\{Z<0\}} \in \mathcal{C} \underset{\overset{\uparrow}{a)}}{\Rightarrow} Z \geq 0$ P-a.s.

c) We have for all $\xi \in L^\infty(\mathcal{F}, P, \mathbb{R}^d)$:

$$E[\xi Z \cdot X_0] = E[\xi X_1 Z]$$

Think of $\xi = (0, \dots, 0, 1\!\!1_A, 0, \dots, 0)$, $A \in \tilde{\mathcal{F}}_0$.

$$\text{So } X_0^{(i)} = E_Q[X_1^{(i)} \mid \tilde{\mathcal{F}}_0]$$

for Q given by $\frac{dQ}{dP} = \frac{Z}{E_P[Z]}$.

d) follows from c). □

We put $\mathcal{Z} := \{ Z \in L^\infty(\mathcal{F}, P) \mid 0 \leq Z \leq 1,$

$P(Z > 0) > 0$ and

$E[ZW] \leq 0 \ \forall W \in \mathcal{C} \}$

Step 4: $\mathcal{C}$ is closed in $L^1(\mathcal{F}, P)$.

We will prove this later.

<u>Step 5:</u> We have by 2° that

$(\mathcal{K} - L^0_+) \cap L^0_+ = \{0\}$. Thus

$\mathcal{C} \cap L^0_+ = (\mathcal{K} - L^0_+) \cap L^1 \cap L^0_+ = \{0\}$.

We prove that (*) $\mathcal{C} \cap L^0_+ = \{0\}$

implies 5.1) $\mathcal{Z} \neq \emptyset$ and

5.2) $\exists\, z \in \mathcal{Z} : P(z > 0) = 1$.

We need for that the Hahn-Banach theorem.

<u>Theorem</u> (Hahn-Banach):

Let $(E, \|\cdot\|)$ be a Banach space, i.e. an $\mathbb{R}$- vector space with norm $\|\cdot\|$, s.t. $(E, \|\cdot\|)$ is complete.

Suppose $\mathcal{B}$ and $\mathcal{C}$ are disjoint closed convex subsets of E such that $\mathcal{B}$ is compact.

Then $\exists\, \ell \in E^* := \{\varphi \in \mathrm{Hom}_{\mathbb{R}}(E, \mathbb{R}) \mid \varphi$ is continuous$\}$ such that

$\sup\{\ell(b) \mid b \in \mathcal{B}\} < \inf\{\ell(c) \mid c \in \mathcal{C}\}$

Proof of 5.1: Take $F \in L^1_+$ non-zero.

Then, by (γ), $F \notin \mathcal{C}$.

Take $\mathcal{B} = \{F\}$ and Hahn-Banach with Radon-Nikodym gives $Z \in L^\infty(\mathcal{F}, P)$ s.t.

$E[ZF] > 0$ and

$E[ZW] \leq C < E[ZF] \quad \forall W \in \mathcal{C}$.

(in particular by Lemma 5.2 (b): $E[ZW] \leq 0$ for all $W \in \mathcal{C}$)

So $Z \in \mathcal{Z}$. $\square$ 5.1.

Proof of 5.2:

$p := \sup \{ P(\{\omega : Z(\omega) > 0\}) \mid Z \in \mathcal{Z} \}$.

Take a sequence $(Z_n)_{n \in \mathbb{N}}$ in $\mathcal{Z}$

such that $P(Z_n > 0) \longrightarrow p$.

Put $Z := \sum_{n=1}^{\infty} \frac{1}{2^n} Z_n$

Then, by Lebesgue convergence theorem, $Z \in \mathcal{Z}$. (exercise).

Assume $P(Z>0)<1$. Then

$\mathbb{1}_{\{Z=0\}} \in L^1_+ \setminus \{0\}$. So

by the proof of 5.1. $\exists\ \tilde{Z} \in$ ~~...~~ $\mathcal{Z}$:

$E[\tilde{Z}\,\mathbb{1}_{\{Z=0\}}]>0$

Then $P(\frac{1}{2}\tilde{Z}+\frac{1}{2}Z>0)>P(Z>0)=p$ ↯

□ 5.2.

Next time, we prove Step 4.

⌈ Small interlude:

L^1 - topology and L^0 - P-a.s. topology.

. On $L^1(\mathcal{F},P)$ we have the topology given by the norm $\|\cdot\|_1$ $(\|F\|_1 := E[|F|]$, $F \in L^1(\mathcal{F},P))$

. On $L^0(\mathcal{F},P)$ the P-a.s. topology is defined as follows: We say $[F_n] \xrightarrow{P\text{-a.s.}} [F]$ iff

$F_n \longrightarrow F$ P-a.s., for $F_n, F \in L^0(\mathcal{F},P)$.

A subset $\mathcal{G}$ of $L^0(\mathcal{F},P)$ is said to be closed w.r.t. the P-a.s. topology if

$\forall\, ([F_n])_{\mathbb{N}}$ in $\mathcal{G}$ such that $[F_n] \xrightarrow{P\text{-a.s.}} [F]$: $[F] \in \mathcal{G}$.

end of lecture 17.03.2022

Lemma 52: Given $F_n \in L^0(\mathcal{F}, P)$, $n \in \mathbb{N}$ such that $\liminf_{n\to\infty} |F_n| < \infty$ P-a.s., then there are $\mathcal{F}$-measurable maps $\sigma_m : \Omega \to \mathbb{N}$, $m \in \mathbb{N}$ such that $(F_{\sigma_m})_{m\in\mathbb{N}}$ converges P-a.s. and $\sigma_m < \sigma_{m+1}$ $\forall \omega \in \Omega$ $\forall m \in \mathbb{N}$.

Proof: a) At first we find $(\tau_n)_{\mathbb{N}}$, $\tau_n : \Omega \to \mathbb{N}$ $\mathcal{F}$-measurable such that $(|F_{\tau_n}|)_{\mathbb{N}}$ converges P-a.s. and $\tau_1 < \tau_2 < \tau_3 < \dots$ $\forall \omega \in \Omega$.

b) Then we put $F := \liminf_{n\to\infty} F_{\tau_n}$

c) Then we find $(\sigma_n)_{\mathbb{N}}$ such that $(F_{\sigma_n})_{\mathbb{N}}$ converges P-almost surely and $\sigma_1 < \sigma_2 < \dots$ $\forall \omega$

To a): $\Lambda(\omega) := \liminf_{n\to\infty} |F_n(\omega)|$

$\Rightarrow$ $P(\Lambda < \infty) = 1$.

On $\{\Lambda = \infty\}$ set $\tau_n(\omega) := n$.

$\tau_1(\omega) := 1 \;, \omega \in \Omega.$

Say $\tau_1 < \tau_2 < \tau_3 < \dots < \tau_n$ are defined. We put on $\{\omega \mid \Lambda(\omega) < \infty\}$:

$$\tau_{n+1}(\omega) := \inf\{m \in \mathbb{N} \mid m > \tau_n(\omega) \text{ and } \underbrace{\big|\,|F_m(\omega)| - \Lambda(\omega)\big|}_{(*)} \leq \frac{1}{n+1}\}$$

Then τ_{n+1} is $\mathcal{F}$-measurable and $|F_{\tau_n(\omega)}| \longrightarrow \Lambda(\omega) \quad \forall \omega \in \Omega$.

<u>T5 b)</u> ok.

<u>T5 c)</u> Same idea as in a). Just replace $(*)$ by $|F_{\tau_m(\omega)}(\omega) - F(\omega)|$

to define $\varepsilon_n : \Omega \longrightarrow \mathbb{N}$.

Put $\sigma_n(\omega) := \tau_{\varepsilon_n(\omega)}(\omega)$. $\square$

We want to prove that $\mathcal{C} = (\mathcal{K} - L^0_+) \cap L^1$ is closed in L^1. For that it is enough to show that $(\mathcal{K} - L^0_+)$ is L^0-closed. Because if $B \subseteq L^0$ is L^0-closed then $B \cap L^1$ is L^1-closed

Proof: $F_n \in B \cap L^1$ $F_n \xrightarrow{L^1} F$ then $F_n \xrightarrow{P} F$ (in probability) $\Rightarrow \exists (n_j) : F_{n_j} \xrightarrow{P\text{-a.s.}} F$

$\Rightarrow F \in B \underset{F \in L^1}{\Rightarrow} F \in B \cap L^1$ $\blacksquare$

Note: The topology in L^1 induced by the L^0-topology is coarser!

There are L^1-closed sets which are not of the form $L^1 \cap B$, $B \subseteq L^0$ closed.

Ex: $F_n(x) = \begin{cases} 1, & 0 \le x \le 1 - \frac{1}{n} \\ n, & 1 - \frac{1}{n} < x \le 1 \end{cases}$ $\{F_n \mid n \in \mathbb{N}, n \geq 2\} \subseteq L^1([0,1], \mathcal{B}([0,1]), \lambda)$ L^1-closed, but not of the form $B \cap L^1$!

We need to prove that $\mathcal{K} - L^0_+$ is closed in $L^0(\mathcal{F}, P)$ w.r.t. P-a.s. topology.

We want to work with a subspace of $L^0(\mathcal{F}_0, P, \mathbb{R}^d)$ for which Y is not redundant

$L^0(\mathcal{F}_0, P, \mathbb{R}^d) = N \oplus N^\perp$

$$N := \{\eta \in L^0(\mathcal{F}_0, P, \mathbb{R}^d) \mid \eta Y = 0 \text{ P-as}\}$$

$$N^\perp := \{\xi \in L^0(\mathcal{F}_0, P, \mathbb{R}^d) \mid \xi\eta = 0 \text{ P-as } \forall \eta \in N\}$$

- $N \cap N^\perp = \{0\}$: $\eta \in N \cap N^\perp \Rightarrow \eta^2 = \sum_{i=1}^d (\eta^{(i)})^2 = 0$ P-as.

  $\Rightarrow \eta = 0$ P-as.

- N and $N^\perp$ are closed under P-as. convergence, because the conditions $\eta Y = 0$ P-as and $\xi\eta = 0$ P-as are closed under P-as. convergence.

- N and $N^\perp$ are convex. (exercise.

- We also have $N + N^\perp = L^0(\mathcal{F}_0, P, \mathbb{R}^d)$

<u>Proof</u>: $(L^2(\mathcal{F}_0, P, \mathbb{R}^d), \langle , \rangle_2)$ is a Hilbert space, in particular we orthogonal projections $\pi := \pi_{N \cap L^2(\mathcal{F}_0, P, \mathbb{R}^d)}$ and $\underbrace{\pi_{N^\perp \cap L^2(\mathcal{F}_0 \dots}}_{=: \pi^\perp}$ ~~onto~~ onto $N \cap L^2$ and $N^\perp \cap L^2$.

$\ulcorner$ $(H, \langle,\rangle)$ a Hilbert space. $A \subseteq H$ closed and convex and $\neq \emptyset$. Then $\forall_{x \in H}$ $\exists$ $\pi_A(x) \in A$ such that $\| \pi_A(x) - x \| = \inf \{ \|y - x\| \mid y \in A \}$ $\lrcorner$

$\rightsquigarrow$ office hours.

Consider the standard basis $\{e_1, e_2, \dots, e_d\}$ of $\mathbb{R}^d$:

$e_i = (0, \dots, \overset{(i)}{1}, \dots, 0)^T$.

Then $e_i \in L^2(\mathcal{F}_0, P, \mathbb{R}^d)$.

$\Rightarrow$ We have

$\eta_i := \pi(e_i)$ and $e_i^\perp := \pi^\perp(e_i)$, $i = 1, \dots, d$.

We have $e_i = \eta_i + e_i^\perp$, because otherwise $\exists$ $\eta \in N$: $P((e_i - \eta_i)\eta > 0) > 0$

and therefore for $C > 0$ big enough we have

$$E\Big[(e_i - \eta_i)\underbrace{\eta \mathbb{1}_{\{|\eta| \le C\} \cap \{(e_i - \eta_i)\eta > 0\}}}_{\in N \cap L^2}\Big] > 0$$

$\lightning$ to $e_i - \eta_i \perp N \cap L^2$ w.r.t. $\langle , \rangle_2$.

Take $\xi \in L^2(\mathcal{F}_0, P, \mathbb{R}^d)$.

$$\xi(\omega) = \underbrace{\sum_{i=1}^{d} \xi^{(i)}(\omega)\, \eta_i(\omega)}_{\in \mathcal{N}} + \underbrace{\sum_{i=1}^{d} \xi^{(i)}(\omega) \cdot e_i^{\perp}(\omega)}_{\in \mathcal{N}^{\perp}}.$$

$(\mathcal{N} + \mathcal{N}^{\perp} = L^0(\mathcal{F}_0, P, \mathbb{R}^d))$ $\square$

Now: Y is not redundant w.r.t. $\mathcal{N}^{\perp}$.

<u>Proof</u> (of $\mathcal{R} - L^0_+$ is closed):

$(W_n)_{\mathbb{N}}$ in $\mathcal{R} - L^0_+$ s.t. $W_n \xrightarrow{P\text{-a.s.}} W \in L^0(\mathcal{F}, P)$

For W_n we have $\xi_n \in \mathcal{N}^{\perp}$ and $U_n \in L^+_0$:

$W_n = \xi_n Y - U_n$.

<u>Case 1:</u> $\liminf_{n \to \infty} |\xi_n|_\infty < \infty$ P-a.s.

$\overset{\text{Lemma 52}}{\Longrightarrow}$ $\exists\ \sigma_n : \Omega \to \mathbb{N}$ $\mathcal{F}_0$-measurable:

$(\xi_{\sigma_n})_{\mathbb{N}}$ converges P-a.s., say to ξ.

$\Rightarrow 0 \leq U_{\sigma_n} = \xi_{\sigma_n} Y - W_{\sigma_n} \xrightarrow{P\text{-a.s.}} \xi Y - W$

$\Rightarrow W \in \mathcal{R} - L^0_+$.

Case 2: $P(A) > 0$ for $A := \{\omega \mid \liminf_{n\to\infty} |\xi_n|_\infty = \infty\}$

We show this is not possible.

$$\zeta_n(\omega) := \begin{cases} \dfrac{\xi_n(\omega)}{|\xi_n(\omega)|_\infty}, & \xi_n(\omega) \neq 0 \in \mathbb{R}^d \\ 0, & \xi_n(\omega) = 0 \in \mathbb{R}^d. \end{cases}$$

$\Rightarrow \liminf_{n\to\infty} |\zeta_n| \leq 1.$

So, by Lemma 52, $\exists (\tau_n)_{\mathbb{N}}$ $\mathcal{F}_0$-measurable

$\tau_1 < \tau_2 < \dots$ P-a.s.

such that $(\zeta_{\tau_n})_{\mathbb{N}}$ converges P-a.s., say to $\zeta \in N^\perp$.

$$\Rightarrow 0 \leq \cancel{\mathbb{1}_A \frac{U_{\tau_n}}{|\xi_{\tau_n}|} = \mathbb{1}_A \frac{\xi_{\tau_n}}{|\xi_{\tau_n}|}}$$

For $\omega \in A$ and n big enough (depending on ω)

$$0 \leq \frac{U_{\tau_n(\omega)}(\omega)}{|\xi_{\tau_n(\omega)}(\omega)|} = \zeta_{\tau_n(\omega)}(\omega)\, Y(\omega) - \frac{W_{\tau_n(\omega)}(\omega)}{|\xi_{\tau_n(\omega)}|}$$

$$\longrightarrow \zeta(\omega)\, Y(\omega)$$

i.e. $\mathbb{1}_A \xi \cdot Y \geq 0$ P-as.

So by $\mathbb{1}_A \xi \in \mathcal{N}^\perp$ we have $\mathbb{1}_A \xi = 0$ Pas.

But $|\xi|_\infty = 1$ on A P-as.

So $P(A) = 0$ ↯.

$\square$ ($\mathcal{R} - L^0_+$ is closed)

$\square$ Theorem 50.

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end of Lecture 11 (22.03.2022)

II Multi period model

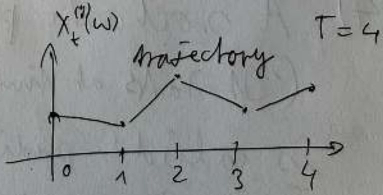
Given $(\Omega, \mathcal{F}, P)$ a probability space
and $d+1$ assets
and times $t = 0, 1, \dots, T$

Def. 5.3:

1) A family $(\mathcal{F}_t)_{t=0,\dots,T}$ of sub-σ-algebras $\mathcal{F}_t$ of $\mathcal{F}$ is called a "filtration" if $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots \subseteq \mathcal{F}_T$.

1) Let $(E, \mathcal{E})$ be a measurable space. A family $(X_t)_{t=0,\dots,T}$ of $\mathcal{F}$-$\mathcal{E}$-measurable maps $X_t : \Omega \to E$ is called a "stochastic process".

Interpretation:
Fix $\omega \in \Omega$:



3) A stochastic process $(X_t)_{t=0,\dots,T}$ for $(\Omega, \mathcal{F})$ is called adapted to a filtration $(\mathcal{F}_t)_{t=\overline{0,T}}$ if X_t is $\mathcal{F}_t$-measurable for all $t=\overline{0,T}$.

<u>Example:</u> Given a stochastic process $(X_t)_{t=\overline{0,T}}$ for $(\Omega, \mathcal{F})$ we can look at the σ-algebras

$$\tilde{\mathcal{F}}_t^X = \sigma(X_0, \dots, X_t), \quad t=\overline{0,T}.$$

("the information ~~possible knowledge~~ that $X_0, \dots, X_t$ can give us.")

4) A stochastic process $(\xi_t)_{t=\overline{1,T}}$ (It starts at time $t=1$!) is called predictable if ξ_t is $\mathcal{F}_{t-1}$-measurable for all $t=\overline{1,T}$.

<u>Setup 54:</u> (market model for multiple periods)

$(\Omega, \mathcal{F}, P)$. $(\mathcal{F}_t)_{t=\overline{0,T}}$ a filtration in $\mathcal{F}$.

$(\underline{S}_t)_{t=\overline{0,T}}$ a stochastic process adapted to $\mathcal{F}_*$.

($S_t^{(i)}$ = value of the ith asset at time t.)

<u>Def. 55:</u> 1) An $\mathcal{F}_*$ predictable $d+1$-dimensional process $(\underline{\xi}_t)_{t\in\overline{1,T}}$ is called "trading strategy".

2) A trading strategy $(\underline{\xi}_t)_{t=\overline{1,T}}$ is called self-financing if for all $t=\overline{1,T-1}$ we have

$$\underline{\xi}_t \underline{S}_t = \underline{\xi}_{t+1} \underline{S}_t .$$

<u>Interpretation 56:</u> • $\xi_t^{(i)}$ – quantity of shares of the ith asset held during the

tth trading period.

- $\underline{\xi}_t \underline{S}_{t-1}$ is the amount invested into the portfolio at time $t-1$.
- $\underline{\xi}_t \underline{S}_t$ is the resulting value of the portfolio at time t.

__Remark 56:__ Let $(\underline{\xi}_t)_{t=\overline{1,T}}$ be a trading strategy. Then $\underline{\xi}_*$ is self-financing iff

$$\underline{\xi}_{t+1}\underline{S}_{t+1} - \underline{\xi}_t \underline{S}_t = \underline{\xi}_{t+1}(\underline{S}_{t+1} - \underline{S}_t)$$

for $t = \overline{1, T-1}$.

We will almost always consider self-financing trading strategies.

__Assumption 57:__ $S_t^0 > 0 \quad \forall t = \overline{0,T}$

The bond never fails.

We get the "discounted price": $X_t^{(i)} := \dfrac{S_t^{(1)}}{S_t^{(0)}}$

and the " — " — value process" $V = (V_t)_t$

$V_0 = \underline{\xi}_1 \underline{X}_0$, $V_t := \underline{\xi}_t \underline{X}_t$, $t = \overline{1,T}$.

and the "discounted gain process": $G = (G_t)_{t=\overline{1,T}}$

$$G_k := \sum_{\ell=1}^{k} \xi_\ell (X_\ell - X_{\ell-1}), \quad k = \overline{1,T}.$$

We put $G_0 := 0$, because at the start there is no gain.

<u>Prop 58:</u> Let $\underline{\xi}$ be a trading strategy. T.a.e.:

1° $\underline{\xi}$ is self-financing

2° $\forall_{t=\overline{1,T-1}}$: $\underline{\xi}_t \underline{X}_t = \underline{\xi}_{t+1} \underline{X}_t$

3° $\forall_{t=\overline{1,T}}$: $V_t = V_0 + G_t$.

<u>Terminology 59:</u> The sum

$$\sum_{t=1}^{T} \underline{\xi}_t (\underline{S}_t - \underline{S}_{t-1}) \text{ is called}$$

"discrete stochastic integral".

Remark 60: Let ξ be self-financing.

Then $\left(\xi_t^{(0)}\right)_{t=\overline{0,T}}$ is determined by V_0 and $(\xi_t)_{t=\overline{1,T}}$.

$$\left(\xi_1^{(0)} = V_0 - \xi_1 X_0; \right.$$

$$\xi_{t+1}^{(0)} - \xi_t^{(0)} = -(\xi_{t+1} - \xi_t) X_t$$

$$\left. t = \overline{1,T-1}. \right)$$

Def. 61: a) A self-financing trading strategy ξ is called an AO if $V_0 \leq 0$ P-a.s. and $V_T \geq 0$ P-a.s. and $P(V_T > 0) > 0$.

b) A market model is called AF if there is no AO.

Prop 62: $\exists$ AO $\Longleftrightarrow$ $\exists\, t \in \{1, \dots, T\}$:

$\exists$ AO for the tth trading period,

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i.e. $\exists \eta \in \mathcal{L}^0(\Omega, \mathcal{F}_{t-1}, P)$:

- $\eta(X_t - X_{t-1}) \geq 0$
- and $P(\eta(X_t - X_{t-1}) > 0) > 0$.

Proof: "$\Leftarrow$" Consider

$$\xi_s := \begin{cases} \eta, & s = t \\ 0, & s \neq t \end{cases}$$

$$V_0 := 0$$

Then $(\xi^{(0)}_t)_{t=1,\dots,T}$ is determined if we want $\underline{\xi}$ to be self-financing.

(Strategy: we wait until the t th trading period and invest then using η. The value of the ~~tth~~ value process at t will be invested into the bond.)

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"$\Rightarrow$" Let $\underline{\xi} = (\xi^0, \xi)$ be an AO.

Put $t := \min\{k \mid V_k \geq 0$ P-a.s. and

$$P(V_k > 0) > 0\}$$

Then we have two cases: (Note $1 \leq t \leq T$)

<u>Case 1:</u> $V_{\cancel{t}-1} = 0$ P-a.s.

<u>Case 2:</u> $P(V_{t-1} < 0) > 0$.

<u>To Case 1:</u> $\xi_t (X_t - X_{t-1})$

$$= \xi_t X_t - \xi_{t} X_{t-1}$$

$$= V_t - V_{t-1} = V_t \geq 0$$

with > 0 with positive probability.

$\Rightarrow$ Take $\eta := \xi_t$.

<u>To Case 2:</u> Take $\eta := \xi_t \cdot \mathbb{1}_{\{V_{t-1} < 0\}}$.

$$\xi_t \mathbb{1}_{\{V_{t-1} < 0\}} (X_t - X_{t-1})$$

$$= \mathbb{1}_{\{V_{t-1} < 0\}} (V_t - V_{t-1}) \geq 0$$

"> 0" pos. prob. □

Def. 63: A process $(M_t)_{t=\overline{0,T}}$ with values in $\mathbb{R}^m$ is called a martingale w.r.t. $(\mathcal{F}_t)_{t=\overline{0,T}}$ and a measure Q if

1) M is adapted
2) $\forall t=\overline{0,T}$: $|M_t|_\infty$ is Q-integrable
3) $\forall t\in\{1,\dots,T\}$: $E_Q[M_t|\mathcal{F}_{t-1}]=M_{t-1}$.

Remark 64: Given a martingal M we have

$M_t = E_Q[M_T \mid \mathcal{F}_t]$ by transitivity.

On the other hand given $N\in L^1(\Omega,\mathcal{F}_T,Q)$ then $(N_t)_{t=\overline{0,T}}$ defined via

$N_t := E[N\mid\mathcal{F}_t]$ is a martingale.

Example 65: a) Let $U_1,U_2,\dots,U_T$ be i.i.d. w.r.t. Q and integrable and $E[U_i]=0\ \forall i$, and $M_0\in\mathbb{R}$

Then $M_t := M_0+\sum_{i=1}^{t} U_i$, $t=0,\dots,T$

is a martingale w.r.t. $(\mathcal{F}_t := \sigma(U_1,\dots,U_t))_{t=\overline{0,T}}$, Q

Proof: a) M_t is $\sigma(U_1,\dots,U_t)$ measuralbe ✓

b) $|M_t| \leq |M_0| + \sum_{i=1}^{t} |U_i|$

and $M_0, U_i \in \mathcal{L}^1(\Omega, \mathcal{F}_T, Q)$

$\Rightarrow M_t \in \mathcal{L}^1(\Omega, \mathcal{F}_t, Q)$

c) Take $A \in \mathcal{F}_t$ and $t \in \{0, \dots, T-1\}$

$$E_Q[M_t \mathbb{1}_A] = E_Q[M_{t-1}\mathbb{1}_A + U_t \mathbb{1}_A]$$

$$= E_Q[M_{t-1}\mathbb{1}_A] + E_Q[U_t \mathbb{1}_A]$$

$$= \quad - \text{"} - \quad + \underbrace{E_Q[U_t]}_{=0} E[\mathbb{1}_A]$$

$\sigma(U_t)$ is independent to $\sigma(U_1,\dots,U_{t-1})$

$$= E_Q[M_{t-1}\mathbb{1}_A].$$

$$\Rightarrow E_Q[M_t | \mathcal{F}_{t-1}] = M_{t-1} \quad \square$$

b) Suppose $(X_t)_{t=\overline{0,T}}$ is a $(Q,(\mathcal{F}_t)_{t=\overline{0,T}})$ martingale with values in $\mathbb{R}^{d+1}$.

Let $(\xi_t)_{t=\overline{1,T}}$ be a self-financing trading strategy and bounded.

<u>Claim</u>: $(G_t)_{t=\overline{0,T}}$ and $(V_t)_{t=\overline{0,T}}$ are $(Q,(\mathcal{F}_t)_{t=\overline{0,T}})$ - martingales.

<u>Proof</u>: (Exercise 8). □

c) $\Omega=[0,1[$, $\mathcal{F}:=\mathcal{B}([0,1[)$

$$\mathcal{F}_t:=\sigma\left(\left\{\left[\frac{k}{2^t},\frac{k+1}{2^t}\right[\;\middle|\; k=0,\dots,2^t-1\right\}\right)$$

Then $\mathcal{F}_0=\{\emptyset,[0,1[\}\subseteq\mathcal{F}_1\subseteq\mathcal{F}_2\subseteq\dots\subseteq\mathcal{F}$ and $\bigcup_{t=0}^{\infty}\mathcal{F}_t=\mathcal{F}$.

Let μ be a probability measure on $(\Omega,\mathcal{F})$ such that $\mu\ll\lambda$. Write $\mu_k:=\mu|_{\mathcal{F}_k}$ and $\lambda_k:=\lambda|_{\mathcal{F}_k}$

Consider $Z:=\frac{d\mu}{d\lambda}$ and $Z_k:=\cancel{\frac{d\mu}{d\lambda}\Big|_{\mathcal{F}_k}}=\frac{d\mu_k}{d\lambda_k}$.

We have $Z_k(\omega) = \sum_{s=0}^{2^k-1} \frac{\mu([\frac{s}{2^k}, \frac{s+1}{2^k}[)}{\frac{1}{2^k}} \cdot 1\!\!1_{[\frac{s}{2^k}, \frac{s+1}{2^k}[}$

<u>Claim:</u> $(Z_k)_{k\in\mathbb{N}_0}$ is an $((\mathcal{F}_k)_{k\in\mathbb{N}_0}, \lambda)$-martingale.

<u>Proof:</u> $|Z_k| \leq 2^k$ and Z_k is $\mathcal{F}_k$-measurable

$\Rightarrow Z_k \in L^1(\Omega, \mathcal{F}_k, \lambda)$

Take $k < l$ and $A \in \mathcal{F}_k$.

$\Rightarrow E_\lambda[Z_k 1\!\!1_A] = E_\mu[1\!\!1_A]$

$= \mu(A) \underset{\substack{\uparrow \\ A \in \mathcal{F}_l}}{=} E_{\mu_l}[1\!\!1_A] = E_{\lambda_l}[Z_l 1\!\!1_A]$ ✓

Thus $E_\lambda[Z_l | \mathcal{F}_k] = Z_k \; \forall_{k<l}$ □

Note: In fact: $Z_k = E_\lambda[Z | \mathcal{F}_k]$

($Z \in L^1(\Omega, \mathcal{F}, \lambda)$, because $Z \geq 0$ and $E_\lambda[Z] = \mu(\Omega) = 1$.)

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<u>Remark 82</u>: We will later see that

- $(Z_k)_{k \in \mathbb{N}_0}$ converges A almost surely

and

- that $Z_k \xrightarrow{A\text{-a.s.}} \underset{\substack{\shortparallel \\ \frac{d\mu}{d\lambda}}}{Z}$ if $(Z_k)_{\mathbb{N}_0}$ converges in L^1.

(later)

<u>Def. 83</u>: Let Q be a measure on $(\mathcal{F}, \Omega)$

Q is called $\mathcal{F}$-trivial if

$\forall_{A \in \mathcal{F}}: \; Q(A) \in \{0, 1\}$

<u>Prop. 84</u>: Let $(\Omega, \mathcal{F}, Q)$ be a probability space such that Q is $\mathcal{F}$-trivial. Then for all $F \in L^1(\Omega, \mathcal{F}, Q)$ we have

$$F = \cancel{E_Q} \; E_Q[F] \quad Q\text{-almost surely.}$$

<u>Proof</u>: $A_c := \{\omega \in \Omega \mid F(\omega) < c\}$

$\Rightarrow Q\left(\bigcap_{c \in \mathbb{Q}} A_c\right) = 0$, because $\operatorname{im}(F) \subseteq \mathbb{R}$

and $Q\left(\bigcup_{k\in\mathbb{N}} A_k\right) = 1$
$= \lim_{k\to\infty} Q(A_k)$.

Let $c_0 := \sup\{c \in \mathbb{R} \mid Q(A_c) = 0\}$

Then $\cdot\ \forall d < c_0:\ Q(A_d) = 0$
(because $\exists\, a \in]d, c_0[: Q(A_a) = 0$)

$\cdot\ \forall e > c_0:\ Q(A_e) = 1$, because of the definition of c_0.

$$\Rightarrow Q(\{\omega\in\Omega \mid F(\omega) \le c_0\}) \underset{\substack{\uparrow \\ \text{P a measure}}}{=} \lim_{n\to\infty} Q\left(\{\omega\in\Omega \mid F(\omega) \le c_0 + \tfrac{1}{n}\}\right)$$
$$= \lim_{n\to\infty} 1 = 1.$$

Thus $F(\omega) = c_0$ Q-a.s. $\square$

<u>Prop 85:</u> Suppose we are given a market model with $\mathcal{F}_T = \mathcal{F}$. Let Q be a probability measure on $(\Omega, \mathcal{F})$ trivial on $\mathcal{F}_0$. Then are equivalent:

1° Q is a martingale measure

for $(X_t)_{t=0,T}$

2° $\forall$ L^∞, self-financing trading strategies ξ : V is a Q-martingale

3° $\forall$ self-financing trading strategies ξ with $E_Q[V_T^-] < \infty$:

V is a Q-matingale

4° $\forall$ self-financing trading strategies ξ with ~~P~~ $V_T \geq 0$ ~~Q~~ a.s. :

$E_Q[V_T] = V_0$

~~Proof~~:

<u>Remark 86;</u> If $P = Q$ satisfies d)

1) Then we say that the maket model satisfies the "efficient market hypothesis".

2) If $Q \approx P$, then 4) implies AF; because an AO needs to satisfy

$$\underset{\substack{\uparrow \\ A\,Q}}{0 \geq} V_0 \overset{d)}{=} E_Q[V_T] \underset{\substack{\uparrow \\ P(V_T > 0) > 0}}{>} 0 \,,$$

which is impossible.

<u>Proof</u> (Prop. 85.)

a) $\Rightarrow$ b) (exercise)

b) $\Rightarrow$ c) From a) follows that every $X_t^{(i)}$ is Q- integrable (why?)

Suppose now $E_Q[V_T^-] < \infty$.

Then $E_Q[V_T | \mathcal{F}_{T-1}]$ is well-defined (a priori not necessarily integrable.)

and by Jensen's inequality we have
$$E_Q[V_{T-1}^-] = E_Q[(E_Q[V_T | \mathcal{F}_{T-1}])^-]$$
$$\leq E_Q[E_Q[(V_T)^- | \mathcal{F}_{T-1}]]$$
$$= E_Q[V_T^-] < \infty.$$

By induction we obtain
$$E[V_t^-] < \infty \quad \forall\, t = \overline{1, T}.$$
(For V_0 we have it already!)

Proof (Prop 85) a) ⇒ b) (Exercise!)

b) ⇒ c). From b) follows that every $X_t^{(1)}$ is Q-integrable (Why?)

We prove c) by induction on T.

$T=1$: Let $(\xi_t)_{t=1}$ be a self-financing trading strategy s.t. $E[V_1^-] < \infty$.

Then ξ_1 is constant because Q is trivial on $\mathcal{F}_0$. b) ⇒ V_0, V_1 is a Q-martingale w.r.t. $\mathcal{F}_0 \subseteq \mathcal{F}_1$, in particular V_1 is Q-integrable.

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$T>1$: Consider $\varsigma_T := \xi_T \mathbb{1}_{\{|\xi_T| \leq a\}}$

and $\varsigma_t = 0$ for $t < T$

and $\tilde{V}_0 = 0$

Then ς is bounded and we obtain that $(\tilde{V}_t)_{t=\overline{0,T}}$ is a Q-martingale by b).

$$\Rightarrow \quad E_Q[\varsigma_T (X_T - X_{T-1}) | \mathcal{F}_{T-1}] = 0.$$

$$\Rightarrow \mathbb{1}_{\{\dots\}} E_Q[V_T \mid \mathcal{F}_{T-1}]$$

$$= E_Q[\mathbb{1}_{\{\dots\}} V_T \mid \mathcal{F}_{T-1}] - E_Q[\xi_T(X_T - X_{T-1}) \mid \mathcal{F}_{T-1}]$$

$$= E_Q[\mathbb{1}_{\{\dots\}} V_{T-1} \mid \mathcal{F}_{T-1}]$$

$$= \mathbb{1}_{\{\dots\}} V_{T-1}$$

$a \nearrow \infty \Rightarrow E_Q[V_T \mid \mathcal{F}_{T-1}] = V_{T-1}$.

Then $E_Q[V_{T-1}^-] = E_Q[(E_Q[V_T \mid \mathcal{F}_{T-1}])^-]$

$$\leq E_Q[E_Q[V_T^- \mid \mathcal{F}_{T-1}]] = E_Q[V_T^-]$$

$\uparrow$ Jensen's inequality

$< \infty$.

c) $\Rightarrow$ d) $\checkmark$ .. $E_Q[V_T] = V_0$, because

$(V_t)_{t=0,T}$ is a Q-martingale by c).

d) $\Rightarrow$ a) ① To show $X_t^{(i)} \in L^1(Q)$

$$\xi_s := \begin{pmatrix} 0 \\ \vdots \\ \mathbb{1}_{\{s \leq t\}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ value in } \mathbb{R}^d$$

$V_0 := X_0^{(i)}$

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$\Rightarrow$ We get a self-financing trading strategy. $\underline{\xi}$

$$V_T = V_0 + \sum_{s=1}^{T} \xi_s (X_s - X_{s-1}) = X_t^{(i)} \geq 0$$

$$\underset{d)}{\Rightarrow} \quad E_Q[X_t^{(i)}] = X_0^{(i)} < \infty$$

② To show $E[X_t^{(i)} | \mathcal{F}_{t-1}] = X_{t-1}^{(i)}$ (*)

$A \in \mathcal{F}_{t-1}$: Put $\eta_s^{(i)} := \mathbb{1}_{\{s<t\}} + \mathbb{1}_{\{s=t\}} \mathbb{1}_{A^c}$

and $V_0 = X_0^{(i)}$. $\eta^{(j)} = 0 \; \forall j \neq i$.

Then $\exists$ self-financing trading strategy $\underline{\eta}$ and we obtain

$$V_T(\underline{\eta}) = X_{t-1}^{(i)} \cdot \mathbb{1}_A + X_t^{(i)} \mathbb{1}_{A^c} \geq 0$$

$$d) \Rightarrow \quad X_0^{(i)} = E_Q[X_{t-1}^{(i)} \mathbb{1}_A] + E_Q[X_t^{(i)}] - E_Q[X_t^{(i)} \mathbb{1}_A]$$

$$\overset{(*)}{\Longrightarrow} \quad E_Q[X_{t-1}^{(i)} \mathbb{1}_A] = E_Q[X_t^{(i)} \mathbb{1}_A]$$

$\square$

<u>Prop 87</u> (FTAP multi-period):

$\mathcal{MM}$ is AF $\Leftrightarrow \mathcal{P} := \{Q \approx P \mid Q$ is a martingale measure for X w.r.t. $\mathcal{F}_t\} \neq \emptyset$

In that case we can find $P^* \in \mathcal{P}$ s.t.

$\frac{dP^*}{dP}$ is bounded.

<u>Proof</u>: "$\Leftarrow$" Prop. 85 (1) $\Rightarrow$ 4)) and Remark 86.

"$\Rightarrow$" Prove by induction:

$$\forall_{t \in \{0, \dots, T-1\}} : \exists \, P^*_{t+1} \approx P \text{ on } \mathcal{F}$$

with bounded density s.t.

$X_t, X_{t+1}, \dots, X_T$ is a P^*_{t+1}-martingale

w.r.t. $\mathcal{F}_t \subseteq \mathcal{F}_{t+1} \subseteq \dots \subseteq \mathcal{F}_T$.

<u>$t := T-1$:</u> $\mathcal{MM}$ is AF $\Longrightarrow$ $\mathcal{MM}$ is AF

on the $(t+1)$th trading period by

by Prop. 62. FTAP for one period solves this case, see Theorem 50.

$0 \le t < T-1$: Prop 62 $\Longrightarrow$ $\mathcal{M}$ is AF

on the $(t+1)$th trading period.

Theorem 50 $\Longrightarrow$ $\exists Q \cancel{P^*_{t+1}} \sim P^*_{t+2}$ on $\mathcal{F}$

a martingale measure for X_t, X_{t+1}

w.r.t. $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$ such that $\dfrac{dQ}{dP^*_{t+2}}$ is

bounded. Put $Z := E_{P^*_{t+2}}\left[\dfrac{dQ}{dP^*_{t+2}} \,\middle|\, \mathcal{F}_{t+1} \right]$

and take P^*_{t+1} with density $\dfrac{dP^*_{t+1}}{dP^*_{t+2}} := Z$.

<u>Then</u> (a) Z is bounded ✓

(b) P^*_{t+1} is a martingale measure

for $X_t, X_{t+1}, \ldots, X_T$ w.r.t.

$\mathcal{F}_t \subseteq \mathcal{F}_{t+1} \subseteq \cdots \subseteq \mathcal{F}_{T-1} \subseteq \mathcal{F}_T$.

Pf: • $E_{P^*_{A+1}}[|X_s^{(i)}|] = E_{P^*_{A+2}}[|X_s^{(i)}| Z]$

$\leq c\, E_{P^*_{A+2}}[|X_s^{(i)}|] < \infty$

$s = A+1, \dots, T$

• $E_{P^*_{A+1}}[|X_A^{(i)}|] = E_{P^*_{A+2}}[|X_A^{(i)}| Z]$

$\underset{\uparrow}{=} E_{P^*_{A+2}}\left[|X_A^{(i)}| \frac{dQ}{dP^*_{A+2}}\right] \underset{\uparrow}{<} \infty$

$|X_A^{(i)}|$ is $\mathcal{F}_A \subseteq \mathcal{F}_{A+1}$ -measurable

Q is a martingale measure for trading period $A+1$.

~~You also could use that $\frac{dQ}{dP}$~~

• For $s = A+2, \dots, T$

$E_{P^*_{A+1}}[X_s^{(i)} | \mathcal{F}_{s-1}] = \dfrac{E_{P^*_{A+2}}[X_s^{(i)} Z | \mathcal{F}_{s-1}]}{E_{P^*_{A+2}}[Z | \mathcal{F}_{s-1}]}$

$\underset{Z \text{ is } \mathcal{F}_{A+1}\text{-measurable}}{\overset{45.3)}{=}} E_{P^*_{A+2}}[X_s^{(i)} | \mathcal{F}_{s-1}] \overset{(IH)}{=} X_{s-1}^{(i)}$

- $s = A+1$

$$E_{P^*_{A+1}}\left[X^{(i)}_{A+1} \mid \tilde{\mathcal{F}}_A\right] = \frac{E_{P^*_{A+2}}\left[X^{(i)}_{A+1} Z \mid \tilde{\mathcal{F}}_A\right]}{E_{P^*_{A+2}}\left[Z \mid \tilde{\mathcal{F}}_A\right]}$$

$$= \frac{E_{P^*_{A+2}}\left[X^{(i)}_{A+1} \frac{dQ}{dP^*_{A+2}} \mid \tilde{\mathcal{F}}_A\right]}{E_{P^*_{A+2}}\left[\frac{dQ}{dP^*_{A+2}} \mid \tilde{\mathcal{F}}_A\right]}$$

$$= E_Q\left[X^{(i)}_{A+1} \mid \tilde{\mathcal{F}}_A\right] = X^{(i)}_A .$$

$\square$

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nd of Lecture 15, 12.4.22

II.2. European contingent claims

Def. 88: An $\mathcal{F}_*$ adapted non-negative process $(C_t)_{t=\overline{0,T}}$ is called an American contingent claim.
An Am. contingent claim is called European " " if $0 = C_0 = C_1 = \dots = C_{T-1}$ P-a.s.. (In this case we just consider $C_T : (\Omega, \mathcal{F}_T) \longrightarrow \mathbb{R}_+ = [0, \infty[$)

If C is adapted w.r.t. ~~$(\sigma(\underline{S}_s, 0 \le s \le t)$~~ $(\sigma(\underline{S}_s \mid 0 \le s \le t))_{t=\overline{0,T}}$ then we call C a derivative.

Def 89: Let C be a European contingent claim w.r.t. $\mathcal{F}_* = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}_T$.
We call T the "expiration date" or "maturity" of C.

<u>Examples 9.10:</u> (a) European call option $(S_T^{(i)}-K)^+$

" put option $(K-S_T^{(i)})^+$

(b) Asian option: based on the average price of the asset $S^{(i)}$

$$S_{Av}^{(i)} := \frac{1}{|\Pi|}\sum_{t\in\Pi} S_t^{(i)}, \text{ where}$$

$$\Pi \subseteq \{0,1,\dots,T\}$$

$$C^{call} = (S_{Av}^{(i)}-K)^+, \quad C^{put} = (K-S_{Av}^{(i)})^+$$

(c) Average strike call/put

$$(S_T^{(i)} - S_{Av}^{i})^+, \quad (S_{Av}^{(i)} - S_T^{(i)})^+$$

(d) "barrier options": some examples

(d1) "digital options":

$$C^{dig} = \begin{cases} 1, & \max\limits_{0\le t\le T} S_t^{(i)} \ge B \\ 0, & \text{else.} \end{cases}$$

d2) $C^{call}_{up\,\&\,out} = \begin{cases} (S^i_T - K)^+, & \text{if } \max\limits_{0\le A\le T} S^i_A < B \\ 0, & \text{else} \end{cases}$

d3) "look back": call/put

lb-call $S^{(i)}_T - \min\limits_{0\le A\le T} S^{(i)}_A$

lb-put $\max\limits_{0\le A\le T} S^{(i)}_A - S^{(i)}_T$

<u>Convention 91:</u> We skip the word European.
Given a contingent claim C we consider the discounted payoff $\frac{C}{S^{(0)}_T} =: H$.

~~Convention~~ <u>Assumption 92:</u>
From now on we assume that the MM is AF, i.e. $\mathbb{P} \neq \emptyset$.
We study the pricing of contingent claims

<u>Def. 93:</u> A contingent claim C is called "replicable" if $\exists$ self-financing trading strategy $\underline{\xi}$, such that $C = \underline{S}_T \cdot \underline{\xi}_T$ P-a.s.

Remark 94 T.f.a. are equivalent:

1° C is replicable

2° $\exists\ \underline{\xi}$ self-financing trading strategy: $H = \underline{\xi}_T \underline{X}_T = V_T = V_0 + \sum_{t=1}^{T} \underline{\xi}_t (\underline{X}_t - \underline{X}_{t-1})$

Prop 95: Suppose P is trivial on $\mathcal{F}_0$.

$\forall$ replicable discounted contingent claims H

$\forall\ P^* \in \mathcal{P}$: $E_{P^*}[H] < \infty$ and for every replication the value process V ~~satisfies~~ satisfies

$$V_t = E_{P^*}[H \mid \mathcal{F}_t] \text{ for all } t = \overline{0,T}$$

(in particular the value process is independent from the replication.)

Proof: It is implied by Prop 85 ($1° \Rightarrow 3°$) $\square$

<u>Assumption 96:</u> From now on we assume that $\mathbb{P}$ is trivial on $\mathcal{F}_0$.

<u>Def 97:</u> (a) $\pi^H \in \mathbb{R}_+$ is called an A.F. price for H, if $\exists$ adapted stoch. process $(X_t^{(d+1)})_{t=\overline{0,T}}$ such that

- $X_0^{(d+1)} = \pi^H$
- $X_t^{(d+1)} \geq 0 \quad \forall_{t=\overline{0,T}}$
- $X_T^{(d+1)} = H$, and
- $\hat{X} = \begin{pmatrix} X^{(1)} \\ \vdots \\ X^{(d+1)} \end{pmatrix}$ is A.F. w.r.t. P.

We denote the set of A.F. prices of H by $\Pi(H)$ and $\Pi_{inf}(H)$, $\Pi_{sup}(H)$ the respective infimum and supremum.

(b) $\pi^C \in \mathbb{R}_+$ is called an A.F. price for C if $\frac{\pi^C}{S_0^{(0)}}$ is an A.F. price for H.

<u>Prop 98:</u> (a) $\Pi(H) \neq \emptyset$

(b) $\Pi(H) = \{ E^*[H] \mid P^* \in \mathcal{P}, E^*[H] < \infty \}$

(c) $\Pi_{inf}(H) = \inf\limits_{P^* \in \mathcal{P}} E^*[H]$

(d) $\Pi_{sup}(H) = \sup\limits_{P^* \in \mathcal{P}} E^*[H]$

<u>Proof:</u> (a) Consider R, a prob. measure on $\mathcal{F}$,

s.t. $\frac{dR}{dP} = \frac{1}{1+|H|}$.

FTAP $\Rightarrow \exists Q \in \mathcal{P} : \frac{dQ}{dR}$ is bounded.

$\Rightarrow E_Q[H] < \infty$. Take

$X_t^{(d+1)} := E_Q[H \mid \mathcal{F}_t], t = \overline{0,T}$

(b) "$\supseteq$" by the argument in (a).

"$\subseteq$" Take $\pi^H \in \Pi(H)$ given

by $(X_t^{(d+1)})_{t=\overline{0,T}}$

$\hat{X}$ is AF $\Rightarrow \exists Q \in \mathcal{P}$ such

that $X^{(d+1)}$ is a Q-martingale

w.r.t. $\mathcal{F}_0$ $\Rightarrow$ $E_Q[X_T^{(d+1)}] = E_Q[X_0^{(d+1)}]$

$$\begin{array}{ccc} E_Q[X_T^{(d+1)}] & = & E_Q[X_0^{(d+1)}] \\ \shortparallel & & \shortparallel \\ E_Q[H] & & \pi^H \end{array}$$

□ (b)

(c) and (d) follow from (a) and (b).

$\cdot\ \Pi_{\inf}(H) = \inf \Pi(H) \underset{(a)}{\overset{\uparrow}{=}} \inf_{P^* \in \mathcal{P}} E^*[H] = \inf_{P^0 \in \mathcal{P}} E^0[H]$

$E^*[H] < \infty$

End of Lecture 16, 14.4.22

$\cdot\ \Pi_{\sup}(H) = \sup \Pi(H) \leq \sup_{P^* \in \mathcal{P}} E^*[H]$

At first: We have $=$ if $E^*[H] < \infty$ for all $P^* \in \mathcal{P}$.

Secondly: Assume $\exists\ P^0 \in \mathcal{P}$: $E^*[H] = \infty$.

To show: $\forall_{c>0} \exists\ \pi \in \Pi(H) : \pi \geq c$

Notation: $a, b \in \overline{\mathbb{R}}$: ($a \wedge b := \min\{a, b\}$
$a \vee b := \max\{a, b\}$)

$c>0$. Consider $n \in \mathbb{N}$ big enough s.t.
$E_*[H \wedge n] \geq c$. (possible by the monotone convergence theorem)

Put $Y_t := E_{P^*}[H \wedge n \mid \mathcal{F}_t]$, $t = \overline{0,T}$.

X, Y is AF because it has P^* as a martingale measure.

$$\mathcal{P}^Y := \{ Q \approx P \mid Q \text{ is a martingale measure for } X, Y \}$$

(e) $\wedge$ (a) $\Rightarrow \exists\, Q \in \mathcal{P}^Y : \quad E_Q[H] < \infty$

$$\Rightarrow \quad E_Q[H] \geq E_Q[H \wedge n] = E_{P^*}[H \wedge n] \geq c. \qquad \square$$

<u>Prop 99</u>: a) H is replicable $\Leftrightarrow |\Pi(H)| = 1$

b) H is not replicable $\Leftrightarrow$

$\Pi_{\inf}(H) < \Pi_{\sup}(H) \Leftrightarrow$

$\Pi(H)$ is an open intervall.

<u>Proof</u>: b) Suppose $\Pi_{\inf}(H) < \Pi_{\sup}(H)$. We have to show that $\Pi(H)$ is an open intervall

$\Pi(H)$ is convex and thus an intervall.

We show that for every $\pi \in \Pi(H)$

$\exists\ \check{\pi}, \hat{\pi} \in \Pi(H)\ :\ \check{\pi} < \pi < \hat{\pi}$.

$\Pi_{\inf}(H) < \Pi_{\sup}(H) \Rightarrow$ H is not replicable.

Take $P^* \in \mathcal{P}$ such that $\pi = E^*[H]$ ~~[illegible]~~.

Consider the process $U_t := E^*[H \mid \mathcal{F}_t]$

$t = 0, \dots, T$.

$$\Rightarrow \quad U_t = U_0 + \sum_{s=1}^{t} (U_s - U_{s-1})$$

$$U_T = H$$

H is not replicable $\Rightarrow \exists\ s \in \{1, \dots, T\}$:

$$U_s - U_{s-1} \notin \mathcal{K}_s := \{ \eta (X_s - X_{s-1}) \mid \eta \in L^0(\Omega, \mathcal{F}_{s-1}, P) \}$$

$\{U_s - U_{s-1}\}$, $\mathcal{C}_s := \mathcal{K}_s \cap L^1(P^*) \subseteq L^1(\dots$

are closed, convex and the first is compact

$$\{U_s - U_{s-1}\} \cap \mathcal{C} = \emptyset.$$

Hahn–Banach $\Rightarrow \exists\ Z \in L^\infty(\Omega, \mathcal{F}_s, P^*)$

such that

$$\sup_{W \in \mathcal{C}_s} E^*[ZW] < E^*[Z(U_s - U_{s-1})]$$

$\mathcal{C}_s$ is an $\mathbb{R}$-vector space and $\sup_{W \in \mathcal{C}_s} E^*[ZW] < \infty$

$$\Rightarrow \quad \sup_{W \in \mathcal{C}_s} E^*[ZW] = 0$$

We can scale Z s.t. $|Z| \leq \frac{1}{3}$.

Put $\hat{Z} := 1 + Z - E^*[Z \mid \mathcal{F}_{s-1}]$ and use $(\Omega, \mathcal{F}, \hat{P})$ with $\frac{d\hat{P}}{dP^*} := \hat{Z}$.

(Note: $\frac{1}{3} \leq \hat{Z} \leq \frac{5}{3}$ and $E^*[\hat{Z}] = 1$)

Then $E_{\hat{P}}[H] = \pi + E^*[HZ] - E^*[H E^*[Z \mid \mathcal{F}_{s-1}]]$

$$\underset{\substack{\uparrow \\ \text{pass to cond.} \\ \text{expectations}}}{=} \pi + E^*[U_s Z] - E^*[U_{s-1} Z]$$

$$> \pi.$$

To show $\hat{P} \in \mathcal{Q}$:

- integrability of $X_k^{(i)}$ ✓ because $X_k^{(i)} \in L^1(P^*)$ and $\frac{\hat{Z}}{Z}$ is bounded
- $k > s$: $\hat{E}[X_k - X_{k-1} \mid F_{k-1}]$

$$= \frac{E^*\left[(X_k - X_{k-1})\frac{\hat{Z}}{Z} \mid F_{k-1}\right]}{E^*\left[\frac{\hat{Z}}{Z} \mid F_{k-1}\right]}$$

$$\overset{\uparrow}{=} \frac{\frac{\hat{Z}}{Z}}{\frac{\hat{Z}}{Z}} \cdot E^*[X_k - X_{k-1} \mid F_{k-1}] = 0$$

$\frac{\hat{Z}}{Z}$ F_{k-1}-measurable

- $k \le s-1$ ✓ because $E^*\left[\frac{\hat{Z}}{Z} \mid F_{k}\right] = 1$
- $k = s$: Here we use that $E^*[Z W] = 0 \ \forall W \in \mathcal{C}_s$

Now consider $\frac{d\check{P}}{dP^\sigma} := 2 - \frac{d\hat{P}}{dP^\sigma}$.

Then $\check{P} \in \mathcal{P}$ and

$$\check{E}[H] = 2\pi - \hat{E}[H]$$

$$= 2\pi - \hat{\pi} = \pi + (\pi - \hat{\pi}) < \pi.$$

□ (openness part of (b))

(a) "$\Rightarrow$" by Prop 95.

"$\Leftarrow$" by the proof of (b) (Check!) □

End of Lecture 17: 26.04.2022.

<u>Question 100:</u> What happens in case of an earlier maturity? ($0 \leq T_0 < T$)

We have $\mathcal{P}_0$ = "set of martingale measures for MM $\left((\underline{S}_t)_{t=0,T_0}, (\mathcal{F}_t)_{t=0,T_0}, P \right)$"

$\mathcal{F}_t \cap \mathcal{F}$

Then $\Pi(H) = \{ E_0^*[H] \mid P_0^* \in \mathcal{P}_0$
$E_0^*[H] < \infty \}$

$\supseteq \{ E^*[H] \mid P^* \in \mathcal{P} \quad E^*[H] < \infty \}$

The elements in $\mathcal{P}$ satisfy more conditions!

Do we have "$=$"? Answer: Yes.

Prop 101: $\forall P_0^* \in \mathcal{P}_0 \quad \exists P^* \in \mathcal{P} : P^*|_{\mathcal{F}_{T_0}} = P_0^*|_{\mathcal{F}_{T_0}}$

Proof: Take $P_0^* \in \mathcal{P}_0$ and $\hat{P} \in \mathcal{P}$.

Put $Z_{T_0} := \dfrac{d(P_0^* | \mathcal{F}_{T_0})}{d(\hat{P} | \mathcal{F}_{T_0})}$ and take

P^* defined via $\dfrac{dP^*}{d\hat{P}} = Z_{T_0}$.

$\Longrightarrow P^* \in \mathcal{P}$ and $P^*|_{\mathcal{F}_{T_0}} = P_0^*|_{\mathcal{F}_{T_0}}$. $\square$

Example 102: Call options

with different maturities. By prop. 101 we only need to consider the "bigger" (more periods) market.

Consider the following setting:

- A locally riskless bond, i.e.

$$S^{(0)}_{(0)} = 1 \quad \text{and} \quad \frac{S^{(0)}_A}{S^{(0)}_{A-1}} = 1 + r_A \geq 1$$

("r_A" is called the "spot rate")

$A = 1, \dots, T$.

- two call options on $S^{(1)}$

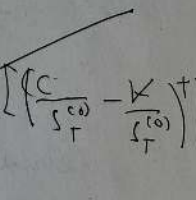
$$C = (S^{(1)}_T - K)^+,$$
$$C_0 = (S^{(1)}_{T_0} - K)^+ \qquad 0 \leq T_0 < T.$$

~~Take $P^* \in \mathcal{P}$.~~

$$E^*\left[\frac{(C-K)^+}{S^{(0)}_T} \Big| \tilde{\mathcal{F}}_{T_0}\right] = E^*\left[\left(\frac{C}{S^{(0)}_T} - \frac{K}{S^{(0)}_T}\right)^+ \Big| \mathcal{F}\right.$$

$$\geq \left(E^*\left[\frac{C}{S^{(0)}_T}\right.\right.$$

~~Jensen's inequality ($x \mapsto (x - \tilde{K})^+$ is convex)~~



Take $P^* \in \mathcal{P}$

$$E^*\left[\frac{(S_T^{(1)}-K)^+}{S_T^{(0)}} \,\middle|\, \mathcal{F}_{T_0}\right] = E^*\left[\left(\frac{S_T^{(1)}}{S_T^{(0)}} - \frac{K}{S_T^{(0)}}\right)^+ \,\middle|\, \mathcal{F}_{T_0}\right]$$

$$\underset{\uparrow}{\geq} \left(E^*\left[\frac{S_T^{(1)}}{S_T^{(0)}} - \frac{K}{S_T^{(0)}} \,\middle|\, \mathcal{F}_{T_0}\right]\right)^+$$

Jensen's inequality ($x \longmapsto x^+$ is convex

)

$$\underset{P^* \in \mathcal{P}}{\overset{\uparrow}{=}} \left(\frac{S_{T_0}^{(1)}}{S_{T_0}^{(0)}} - E^*\left[\frac{K}{S_T^{(0)}} \,\middle|\, \mathcal{F}_{T_0}\right]\right)^+$$

$$= \left(\frac{S_{T_0}^{(1)}}{S_{T_0}^{(0)}} - \frac{K}{S_{T_0}^{(0)}} \underbrace{E^*\left[\frac{S_{T_0}^{(0)}}{S_T^{(0)}} \,\middle|\, \mathcal{F}_{T_0}\right]}_{\leq 1}\right)^+$$

$$= \frac{C_0}{S_{T_0}^{(0)}}.$$

Thus the price is monotone in T, i.e.

$$E_{P^*}\left[\frac{C}{S_T^{(0)}}\right] \geq E_{P^*}\left[\frac{C_0}{S_{T_0}^{(0)}}\right].$$

(Not surprising: Even if C is at T_0 out of money, it can be still in the money at T.

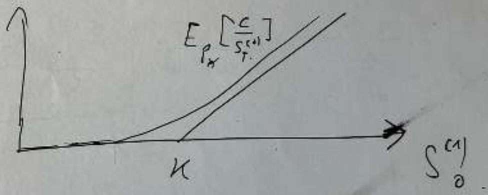
Consider $T_0 = 0$: $E^*\left[\frac{C}{S^{(0)}_T}\right] \geq (S^{(1)}_0 - K)^+$.

$\cdot\ (S^{(1)}_0 - K)^+$ "intrinsic value"

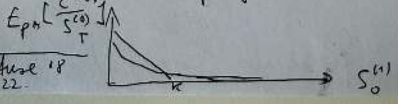
$\cdot\ E^*\left[\frac{C}{S^{(0)}_T}\right] - (S^{(1)}_0 - K)^+$

"time value".

price of a call option as a map depending on $S^{(1)}_0$.



For the put option this is a bit different. A positive return for the bond (and therefore also for $S^{(1)}$ by A.F.) shrinks the payoff at a later time



Finish Lecture 18
28.04.2022.

II 3. Complete multi-period market models

Def. 103: An A.F. multi-period market model is called complete if every European contingent claim is replicable (we also say attainable instead of replicable)

Theorem 104: An A.F. $\underbrace{\text{MM}}_{\text{with } P \text{ trivial on } \mathcal{F}_0}$ is complete

$\iff |\mathcal{P}_{\mathcal{F}_T}| = 1$. $\left(\mathcal{P}_{\mathcal{F}_T} := \{Q|_{\mathcal{F}_T} \mid Q \in \mathcal{P}\right.$

Proof "$\Rightarrow$" $A \in \mathcal{F}_T \Rightarrow H = \mathbb{1}_A \geq 0$ is replicable. Take $Q_1, Q_2 \in \mathcal{P}_{\mathcal{F}_T}$.

Prop. 95 $\Rightarrow E_{Q_1}[H] = E_{Q_2}[H]$ (We use P trivial on $\mathcal{F}_0$)

$\Rightarrow Q_1(A) = Q_2(A)$

"$\Leftarrow$" $|\mathcal{P}_{\mathcal{F}_T}| = 1 \Rightarrow |\pi(H)| = 1$ by Prop 98

Prop. 99(a) $\Rightarrow$ H is attainable.

$\square$

Remark 105: Suppose the MM is complete. Then

$$\dim_{\mathbb{R}} L^0(\Omega, \mathcal{F}_T, P) \leq (1+d)^T.$$

(If P is trivial on $\mathcal{F}_0$)

Proof: $T=1$: ✓

$T>1$: $H \in L^0(\Omega, \mathcal{F}_T, P)$. $H \geq 0$.

is attainable.

$$\Rightarrow \quad H = V_{T-1} + \xi_T (X_T - X_{T-1})$$

for some $\xi_T \in L^0(\Omega, \mathcal{F}_{T-1}, P|_{\mathcal{F}_{T-1}}, \mathbb{R}^d)$

$$V_{T-1} + \xi_T(-X_{T-1}) \in L^0(\Omega, \mathcal{F}_{T-1}, P)$$

and

$$\xi_T X_T \in \sum_{i=1}^{d} L^0(\Omega, \mathcal{F}_{T-1}, P)\, X_T^{(i)}$$

H arbitrary $\Rightarrow L^0(\Omega, \mathcal{F}_T, P) = \sum_{i=0}^{d} L^0(\Omega, \mathcal{F}_{T-1}, P)\, X_T^{(i)}$ □

We can characterize completeness by a not obvious property of the set of martingale measures:

$\mathcal{Q} := \{ Q \text{ a martingale measure on } \mathcal{F}_T \}$

$\mathcal{P} = \{ Q \in \mathcal{Q} \mid Q \approx P \}$

Def. 106: Let R be a convex non-empty subset of an $\mathbb{R}$-vector space.

$\mathcal{E}xt(R) =$ "set of extremal elements of R".

An element x of R is called extremal if

$\forall_{\substack{y, z \in R \\ y \neq z}} \forall_{\varepsilon \in [0,1]} : \left(x = \varepsilon y + (1-\varepsilon) z \Rightarrow \varepsilon \in \{0,1\} \right)$

Prop. 107: Suppose $P^* \in \mathcal{P}$ and P trivial on $\mathcal{F}_0$. T.a.e.:

1° $\mathcal{P} = \{P^*\}$

2° $P^* \in \mathcal{E}xt(\mathcal{Q})$

3° $P^* \in \mathcal{E}xt(\mathcal{P})$

(martingale representation property)

4° $\forall (M_t)_{t=\overline{0,T}}$ $(P^*, (\mathcal{F}_t)_{t=\overline{0,T}})$ - martingale $\exists (\xi_t)_{t=\overline{1,T}}$ predictable:

$$M_t = M_0 + \sum_{s=1}^{t} \xi_s (X_s - X_{s-1})$$

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<u>Proof:</u> $1° \Rightarrow 2°$ If $P^* = \lambda Q_1 + (1-\lambda) Q_2$ with $\lambda \in]0,1[$ and $Q_i \in \mathcal{Q}$.

$\Rightarrow$ $Q_i \ll P^*$, $i=1,2$

$\Rightarrow$ $P_i := \frac{1}{2} Q_i + \frac{1}{2} P^* \approx P$

$\Rightarrow$ $P_1, P_2 \in \mathcal{P} \Rightarrow P_1 = P_2 = P^* = Q_1 = Q_2$

$2° \Rightarrow 3°$ ✓

$3° \Rightarrow 1°$ Assume $\exists \hat{P} \in \mathcal{P} \setminus \{P^*\}$.

W.l.o.g. we can assume that $\frac{d\hat{P}}{dP^*}$ is bounded, say by $C > 0$.

[If not take $\tilde{P} \in \mathcal{P} \setminus \{P^*\}$ and

$A \in \mathcal{F}_T : \tilde{P}(A) \neq P^*(A)$ and

consider $X_t^{(d+1)} = E_{\tilde{P}}[\mathbb{1}_A | \mathcal{F}_t]$

$\Rightarrow \exists$ martingale measure $\hat{P}$ for

$X^{(0)}, \dots, X^{(d+1)}$ and with $\hat{P} \approx P^*$

such that $\frac{d\hat{P}}{dP^*}$ is bounded.]

Now take $\varepsilon > 0$, s.t. $\varepsilon < \frac{1}{c}$.

$$\frac{dP'}{dP^*} = 1 + \varepsilon - \varepsilon \frac{d\hat{P}}{dP^*} \geq 0$$

$$\Rightarrow \quad P^* = \frac{\varepsilon}{1+\varepsilon} \hat{P} + \frac{1}{1+\varepsilon} P' \ \lightning$$

$4° \Rightarrow 1°$ Take $A \in \mathcal{F}_T$ and $Q \in \mathcal{P}$

$\left(E_{P^*}[\mathbb{1}_A | \mathcal{F}_t] \right)_{t=0}^{T}$ is a martingale $\overset{4°}{\Rightarrow}$ It is attainable

$\overset{\text{Prop. 95}}{\Longrightarrow} E_Q[\mathbb{1}_A] = E_{P^*}[\mathbb{1}_A] \Rightarrow Q(A) = P^*(A)$

— $1^{\circ} \Rightarrow 4^{\circ}$ $(M_t)_{t=\overline{0,T}}$ $(P^*, (\mathcal{F}_t)_{t=\overline{0,T}})$ —

martingale. $M_T = M_T^+ - M_T^-$

M_T^+, M_T^- replicable.

$\Rightarrow \exists (\xi_t)_{t=\overline{1,T}}, (\varsigma_t)_{t=\overline{1,T}}$ predictable:

$$M_T^+ = E_{P^*}[M_T^+] + \sum_{t=1}^{T} \xi_t (X_t - X_{t-1})$$

$$M_T^- = E_{P^*}[M_T^-] + \sum_{t=1}^{T} \varsigma_t (X_t - X_{t-1})$$

Note that the value processes for M_T^+ and M_T^- are P^*-martingales by Prop 85.

$(1^{\circ} \Rightarrow 2^{\circ})$

$$\Rightarrow M_T = M_0 + \sum_{t=1}^{T} (\xi_t - \varsigma_t)(X_t - X_{t-1})$$

$$M_t = E_{P^*}[M_T \mid \mathcal{F}_t] = M_0 + \sum_{s=1}^{t} (\xi_s - \varsigma_s)(X_s - X_{s-1})$$

$$+ \sum_{u=t+1}^{T} (\xi_u - \varsigma_u) \underbrace{E_{P^*}[X_u - X_{u-1} \mid \mathcal{F}_t]}_{=0}$$

$\square$

II 4. The binomial model.

(The Cox – Ross – Rubinstein model 1979)

$d = 1$; $S_t^{(0)} = (1+r)^t$ $\quad r > -1$. r is constant.

$S = S^{(1)}$ with return $R_t = \dfrac{S_t - S_{t-1}}{S_{t-1}} \in \{a, b\}$

$-1 < a < b$, $a, b \in \mathbb{R}$.

So we have for the t th trading period

$$\begin{tikzcd}[column sep=large] & S_t = S_{t-1}(1+b) \\ S_{t-1} \arrow[ur, no head] \arrow[dr, no head] & \\ & S_t = S_{t-1}(1+a) \end{tikzcd}$$

$\Omega = \{1, -1\}^T$ "up – down".

$Y_t : \Omega \longrightarrow \{1, -1\}$ $\quad Y_t(\omega) = \omega_t$.

Then $R_t(\omega) = \begin{cases} b, & Y_t(\omega) = 1 \\ a, & Y_t(\omega) = -1 \end{cases}$

$$= a \frac{(1 - Y_t(\omega))}{2} + b \frac{(1 + Y_t(\omega))}{2}.$$

— 108 —

and we can write $S_t = S_0 \prod_{s=1}^{t} (1+R_s)$

$$X_t = S_0 \prod_{s=1}^{t} \frac{(1+R_s)}{(1+r)}.$$

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filtration: $\mathcal{F}_t = \sigma(S_0, S_1, \ldots, S_t)$

$$= \sigma(X_0, \ldots, X_t)$$

$$= \sigma(Y_0, \ldots, Y_t).$$

$t \in \{0, \ldots, T\}$. In particular $\mathcal{F}_T = \mathcal{P}(\Omega)$.

Let P be any probability measure on $(\Omega, \mathcal{F}_T)$

with $P(\{\omega\}) > 0 \ \forall \omega \in \Omega$.

The model for the stock:

$$X_t = S_0 \prod_{s=1}^{t} \frac{(1+R_s)}{(1+r)}$$ with $R_s(\omega) \in \{a, b\}$ (random var.)

and $r > -1$

is called the "Cox - Ross - Rubinstein model".

Prop 108: CRR is AF $\Leftrightarrow$ $a < r < b$.

If CRR is AF then it is complete and $\mathcal{P} = \{P^*\}$ satisfying

- $R_1, \dots, R_T$ are i.i.d and
- ~~$P^*(R_1 = b) = \frac{b-r}{b-a}$~~.

$$P^*(R_1 = b) = \frac{r-a}{b-a} =: p^*$$

Proof: "$\Leftarrow$" Exercise! (Note P^* above is the product measure $P^* = \overbrace{\mu \times \dots \times \mu}^{T}$ with $\mu : \mathcal{P}(\{+1, -1\}) \longrightarrow [0,1]$ given by $\mu(\{1\}) = p^*$

"$\Rightarrow$" Let CRR be AF and take $Q \in \mathcal{P}$.

Then for $t \in \{1, \dots, T\}$:

$$E_Q[X_t \mid \mathcal{F}_{t-1}] = X_{t-1} \, E_Q\left[\frac{1+R_t}{1+r} \,\middle|\, \mathcal{F}_{t-1}\right]$$

$$\overset{=}{\uparrow} \quad X_{t-1} \left(\frac{1+a}{1+r} E_Q[\mathbb{1}_{\{R_t=a\}} | \mathcal{F}_{t-1}] \right.$$

$$1 = \mathbb{1}_{\{R_t=a\}} + \mathbb{1}_{\{R_t=b\}}$$

$$\left. + \frac{1+b}{1+r} E_Q[\mathbb{1}_{\{R_t=b\}} | \mathcal{F}_{t-1}] \right)$$

$$\Rightarrow \quad E_Q[\mathbb{1}_{\{R_t=b\}} | \mathcal{F}_{t-1}]$$

$$= \frac{r-a}{b-a}.$$

In particular R_t and $\mathcal{F}_{t-1}$ are independent and

$$Q(R_t = b) = \frac{r-a}{b-a}. \quad (*)$$

$P \approx Q \Rightarrow \quad a < r < b.$

(**): By induction: $R_1, \dots, R_T$ are i.i.d. w.r.t. Q

Every singleton $\subseteq \Omega$ is of the form $\{\omega \in \Omega \mid R_1(\omega) = r_1, \dots, R_T(\omega) = r_T\}$

for some $(r_1, \dots, r_T) \in \{a, b\}^T$.

Thus by (*) and (**): $|\mathcal{P}| = 1$. $\square$

In AFCRR we can study the value process of a contingent claim H and we can compute the replication explicitely. This is what we do in the remaining part of II.4.

$$H = h(S_0, \dots, S_T) = V_T = v_T(S_0, \dots, S_T)$$

$$V_t = E_{P^*}[v_T(S_0, \dots, S_T) \mid \mathcal{F}_t]$$

$$\underset{\substack{\uparrow \\ \text{atomic} \\ \text{case}}}{=} v_t(S_0, \dots, S_t) \quad \text{for some Borel measurable } v_t : \mathbb{R}^{t+1} \to \mathbb{R}.$$

$$t = 0, \dots, T.$$

We can compute a choice of $(v_t)_{t=\overline{0,T}}$ inductively: $v_T = h$.

$\cancel{v_t(s_0, \dots, s_t) =}$

$$v_t(s_0, \dots, s_t) = E_{P^*}[V_{t+1} \mid \mathcal{F}_t](\omega)$$

(for ω with $S_0(\omega) = s_0, S_1(\omega) = s_1, \dots, S_t(\omega) = s_t$)

$$\underset{\substack{\uparrow \\ \text{atomic case} \\ \text{for conditional} \\ \text{expectation.}}}{=} E_{p^*}\left[v_{t+1}(s_0, s_1, s_2, \dots, s_t, S_{t+1}) \,\middle|\, S_0 = s_0, S_1 = s_1, \dots S_t = s_t \right]$$

$$= p^* v_{t+1}(s_0, s_1, \dots, s_t, s_t(1+b)) + (1-p^*) v_{t+1}(s_0, s_1, \dots, s_t, s_t(1+a))$$

Example 109: (a) $H := h(S_T)$, $\hat{a} := a+1$, $\hat{b} := b+1$

$$\Rightarrow \quad V_t(\omega) = v_t(S_t(\omega)).$$

$$v_t(s_t) = \sum_{k=0}^{T-t} h(s_t \hat{a}^k \hat{b}^{T-t-k}) p^{*T-t} \cdot (1-p^*)^k \cdot \binom{T-t}{k}.$$

$$\Rightarrow \quad V_0 = \sum_{k=0}^{T} h(S_0 \hat{a}^k \hat{b}^{T-k}) p^{*T-k} (1-p^*)^k \cdot \binom{T}{k}$$

End Lecture 21 10.5.22

(b) $H = \frac{C^{call}}{(1+r)^T} = \frac{(S_T - K)^+}{(1+r)^T} = h(S_T)$

$\Rightarrow \Pi(H)$

$$\Pi(H) = \frac{1}{(1+r)^T} \sum_{k=0}^{T} \binom{T}{k} p^{*k} (1-p^*)^{T-k} \left(S_0 \hat{a}^{T-k} \hat{b}^k - K\right)^+$$

$\left(h(s) := \frac{(s-K)^+}{(1+r)^T} \right)$

(c) Running maximum

$$M_t = \max_{0 \le k \le t} S_k$$

$$H(\omega) = h(S_T(\omega), M_T(\omega))$$

The processes $(S_t)_{t=\overline{0,T}}$ and $(M_t)_{t=\overline{0,T}}$ have an interesting property, which already has an effect on computing the value process for H.

Note that $\left(\frac{S_{T-t}}{S_0}, \frac{M_{T-t}}{S_0} \right)$

has the ~~#~~ same distribution as $\left(\frac{S_T}{S_t}, \frac{M_T}{S_t} \right)$

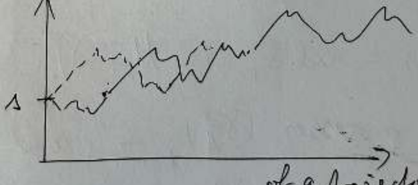
We have $V_t(\omega) = E_{P^*}[H \mid \mathcal{F}_t]$

$\overset{(*)}{=} v_t(S_t(\omega), M_t(\omega))$ ("weak Markov-property")

$$v_t(s_t, m_t) = E_{P^*}\left[h\left(s_t \frac{S_{T-t}}{S_0}, m_t \vee \frac{s_t M_{T-t}}{S_0} \right) \right]$$

(*) is part of the "Markov property of (S_t, M_t)"

Visualize:



A typical behaviour of a trajectory does not depend on the time but only on the starting value of the trajectory.

i.e. a trajectory starting at 0 with start s_0 occurs with the same probability as trajectory starting at t if the S-value at t is s_0.

<u>Def 110:</u> An integrable adapted process Y is called a weak Markov-process w.r.t. P if $\forall f : \mathbb{R} \to \mathbb{R}$ bounded and Borel measurable $\forall\, 0 \leq s < t \leq T$:

$$E[f(Y_t) \mid \mathcal{F}_s] = E[f(Y_t) \mid \sigma(Y_s)].$$

We now want to hedge a contingent claim H in AF CRR.

<u>Prop 111:</u> Suppose the CRR is AF and H is a discounted contingent claim:

$$H(\omega) = h(S_0(\omega), \dots, S_T(\omega)).$$

A replicating self-financing trading strategy $\underline{\xi} = (\xi^0, \xi)$ is given by

$$\xi_t(\omega) = \Delta_t(S_0(\omega), \ldots, S_{t-1}(\omega))$$

with

$$\Delta_t(s_0, \ldots, s_{t-1}) := (1+r)^t \frac{v_t(s_0, \ldots, s_{t-1}, s_{t-1}\hat{b}) - v_t(\ldots, s_{t-1}\hat{a})}{s_{t-1}\hat{b} - s_{t-1}\hat{a}}$$

$$= (1+r)^t \frac{v_t(s_0, s_1, \ldots, s_{t-1}, s_{t-1}\hat{b}) - v_t(s_0, \ldots, s_{t-1}, s_{t-1}\hat{a})}{s_{t-1}\hat{b} - s_{t-1}\hat{a}}.$$

(Δ_t is called the "discrete derivative")

This hedging method is called the "delta - hedge".

<u>Proof</u>: Exercise! $\square$

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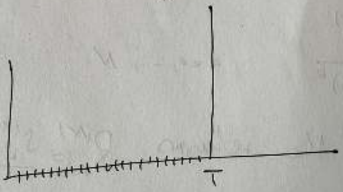
end of Lecture 22: 12.5.22

II 5. Convergence to the Black-Scholes model

1973 formula for the price of a call option given by Black-Scholes.

MM. with many trading periods.

<u>Here</u>: T is a <u>fixed time</u> (not the number of periods)



$[0,T]$ with N periods

$0, \frac{T}{N}, \frac{2T}{N}, \dots, \frac{N-1}{N}T, T.$

Assumption: riskless numeraire with return $r_N > -1$

Note we have $(1+r_N)^N \xrightarrow{N\to\infty} e^{rT}$

$$\iff N r_N \xrightarrow{N\to\infty} rT$$

One risky asset

$\left(S_k^{(N)}\right)_{k=0,\dots,N}$ $S_0^{(N)} = S_0$ constant and independent of N.

with respect to the filtration

$\mathbb{F}^{(N)}$: $\mathcal{F}_k^{(N)} = \sigma\left(S_0^{(N)}, \dots, S_k^{(N)}\right)$

on $\Omega^{(N)}$.

- a martingale measure P_N^* for

$X_k^{(N)} := \dfrac{S_k^{(N)}}{(1+r_N)^k}$, $k=0,\dots,N$.

We assume that the returns $R_k^{(N)} := \dfrac{S_k^{(N)} - S_{k-1}^{(N)}}{S_{k-1}^{(N)}}$

$k=1,\dots,N$ satisfy

- $R_1^{(N)}, \dots, R_N^{(N)}$ are P_N^* independent
- $-1 < \alpha_N \leq R_k^{(N)} \leq \beta_N$ for $k=1,\dots,N$

  such that $\lim\limits_{N\to\infty} \alpha_N = \lim\limits_{N\to\infty} \beta_N = 0$.

(Note: $\lim\limits_{N\to\infty} r_N = 0$.)

$$(*) \quad \cdot \ \frac{1}{T} \sum_{k=1}^{N} \operatorname{var}_{P_N^*}\left(R_k^{(N)}\right) \xrightarrow{N \to \infty} \sigma^2 > 0$$

Interpretation of $(*)$:
The ~~variance of the~~ return of the discounted asset is the compound return over the N trading periods:

$$\left(\prod_{k=1}^{N}\left(1+\left(\frac{1+R_k^{(N)}}{1+r_N}-1\right)\right)\right)-1$$

So we obtain the variance

$$\left(\prod_{k=1}^{N}\left(1+\operatorname{var}_{P_N^*}\left(R_k^{(N)}\right)\frac{1}{(1+r_N)^2}\right)\right)-1$$

And for $N \to \infty$ we want that to ~~go to~~ converge to $e^{\sigma^2 T}-1$.

Theorem 112 (5.54)

$$\left(P_N^{\sigma}\right)^{S_N^{(N)}} \xrightarrow{w} \log N\left((\log S_0) + rT - \frac{\sigma^2}{2}T, (\sigma\sqrt{T})^2\right)$$

a "log-normal distribution", i.e. it is the distribution of the variable

$$S_T = S_0 \exp\left(\sigma W_T + rT - \frac{\sigma^2}{2}T\right)$$

where $W_T \sim N(0,T)$.

(In the logarithmic scaling the values are normally distributed.)

Explanation: (a) $\xrightarrow{w}$ means weak convergence (See A.5.), i.e. given a metric space (M,d) and finite measures $(\mu_n)_{\mathbb{N}}, \mu$ on $\mathcal{B}(M,d)$ (Borel σ-algebra of (M,d))

Then we define $\mu_n \xrightarrow{w} \mu$ ("μ_n converges weakly to μ") if for all

$f \in C_b(M,d) := \{g : M \to \mathbb{R} \mid g \text{ continuous and bounded}\}$

we have $\int_M f \, d\mu_n \xrightarrow{n \to \infty} \int_M f \, d\mu =: \Phi(f,\mu)$.

End of Lecture 23, 17.5.22

(b) We have a topology on

$\text{meas}(M,d) := \{\mu : \mathcal{B}(M,d) \to \mathbb{R}_+ \mid$

μ a measure$\}$, defined as the coarsest topology such that all maps $\Phi(f,-)$ are continuous, $f \in C_b(M,d)$, the "weak topology" τ_w.

(c) <u>Exercise</u>: Let $(\mu_n)_{\mathbb{N}}$ be a sequence in $\text{meas}(M,d)$ and $\mu \in \text{meas}(M,d)$. Then are equivalent:

1° $\mu_n \xrightarrow{w} \mu$

2° $\mu_n \xrightarrow{\tau_w} \mu$.

Before we prove Theorem 112, we look at the example of a CRR model in this setting.

<u>Example 113:</u> CRR – model

$$r_N = \frac{rT}{N}$$

$$\hat{a}_N = 1 + a_N = e^{-\sigma\sqrt{\frac{T}{N}}}$$

$$\hat{b}_N = 1 + b_N = e^{\sigma\sqrt{\frac{T}{N}}}$$

$R_k^{(N)}$, $1 \leq k \leq N$, and P_N^* are given by the model.

<u>Claim</u>: $(*)$ is satisfied and $-1 < a_N \lneq r_N \lneq b_N$ for N big enough.

- To get $-1 < a_N \lneq r_N \lneq b_N$ we consider the inequalities

$$e^{-\sigma x} \lneq 1 + x^2 r \lneq e^{\sigma x} \text{ for small}$$

positive x. Those inequality hold because we get for the tangent lines at 0 the slops:

$$\left.\frac{d e^{-\sigma x}}{dx}\right|_{x=0} = -\sigma, \quad \left.\frac{d(1+x^2 r)}{dx}\right|_{x=0} = 0$$

$$\left.\frac{d e^{\sigma x}}{dx}\right|_{x=0} = \sigma > 0$$

- We now prove (*). ~~for N big enough.~~

$$\sum_{k=1}^{N} \mathrm{var}_{P_N^*}\left(R_k^{(N)}\right) \underset{\substack{\uparrow \\ R_1^{(N)},\dots,R_N^{(N)} \text{ are i.i.d.}}}{=} N\left(\mathrm{var}_{P_N^*}\left(R_1^{(N)}\right)\right)$$

$$\underset{\substack{\uparrow \\ P_N^*(R_1^{(N)}=b_N) = \frac{r_N - a_N}{b_N - a_N} =: p_N^*}}{=} N \cdot \left(\left(p_N^* b_N^2 + (1-p_N^*)\, a_N^2\right) - r_N^2 \right)$$

$$\longrightarrow \sigma^2 T, \text{ because}$$

- $$\lim_{N\to\infty} \frac{r_N - a_N}{b_N - a_N} = \lim_{x\downarrow 0} \frac{rx^2 - (e^{-\sigma x} - 1)}{e^{\sigma x} - e^{-\sigma x}}$$

$$\underset{\substack{\uparrow \\ \text{l'Hospital}}}{=} \frac{\sigma}{2\sigma} = \frac{1}{2}.$$

- $$\lim_{N\to\infty} N r_N^2 = (rT)^2 \lim_{N\to\infty} \frac{1}{N} = 0$$

- $$\lim_{N\to\infty} N b_N^2 = T \lim_{x\downarrow 0} \left(\frac{e^{\sigma x} - 1}{x} \right)^2 = \sigma^2 T$$

$$\lim_{N\to\infty} N a_N^2 = \sigma^2 T.$$

We use the Proposition from the appendix for the proof of Theorem 112.

Proposition 114: (Theorem A.21) "Central Limit Theorem"

Let $Y_1^{(N)}, \dots, Y_N^{(N)}$ be random ~~vars~~ ~~vars~~ ~~vari~~ variables on (Ω_N, P_N). Suppose that they satisfy:

- They are independent, i.e. $Y_1^{(N)}, \dots, Y_N^{(N)}$ are independent.
- $\exists\, \gamma_N > 0$ a real number: $\max_{1 \le k \le N} |Y_k^{(N)}| \le \gamma_N$ P_N-almost surely, and we can choose such a sequence $(\gamma_N)_{N \in \mathbb{N}}$ such that $\lim_{N \to \infty} \gamma_N = 0$.
- $\sum_{k=1}^{N} E_{P_N^*}[Y_k^{(N)}] \xrightarrow{N \to \infty} m \in \mathbb{R}$
- $\sum_{k=1}^{N} \operatorname{var}(Y_k^{(N)}) \xrightarrow{N \to \infty} \sigma^2$

Then $\sum_{k=1}^{N} Y_k^{(N)} \xrightarrow{w} N(m, \sigma^2)$.

Proof of Theorem 112: We analyse the weak convergence of $S_N^{(N)} = S_0 \prod_{k=1}^{N}(1+R_k^{(N)})$.

Step 1: on $]-1,1]$ we have

$$\ln(1+x) = \sum_{n=1}^{\infty}\left(-\frac{(-x)^n}{n}\right) = x - \frac{x^2}{2} + \rho(x)\,x^2$$

with $\rho(x) = -\sum_{n=1}^{\infty} \frac{(-x)^n}{n+2}$.

(Note that the series for ρ converges on $]-1,1]$ by Leibniz and the ratio test. Thus ρ is continuous on $]-1,1]$ by the M-test and Abel.)

For $-1 < \alpha \le \beta \le 1$ we define

$$\delta(\alpha,\beta) := \max\{|\rho(x)| \mid x \in [\alpha,\beta]\}$$

Then by continuity of ρ at 0 we get

$$\delta(\alpha,\beta) \xrightarrow{(\alpha,\beta)\to(0,0)} 0$$

Step 2: We want to show

$$\ln S_N^{(N)} - \ln S_0 \xrightarrow{w} N\left(rT - \frac{\sigma^2}{2}T,\ \sigma^2 T\right)$$

$$\underset{\left(\sum_{k=1}^{N}\left(R_k^{(N)} - \frac{1}{2}\left(R_k^{(N)}\right)^2\right)\right) + \Delta_N}{\shortparallel}$$

end of Lecture 24. 19.5.22.

We check the preconditions of Prop. 114. for

$$Y_k^{(N)} := R_k^{(N)} - \frac{1}{2}\left(R_k^{(N)}\right)^2$$

· $$E_{P_N^*}\left[\sum_{k=1}^{N} Y_k^{(N)}\right] = \sum_{k=1}^{N}\left(r_N - \frac{1}{2}\left(var\left(R_k^{(N)}\right) + r_N^2\right)\right)$$

$$\longrightarrow Tr - \frac{1}{2}\sigma^2 T$$

(because $N r_N^2 \xrightarrow{N\to\infty} 0$.)

· $$\left|Y_k^{(N)}\right| \leq \frac{(|\alpha_N| + |\beta_N| \cancel{\ })^2 + (|\alpha_N| + |\beta_N|)}{2}$$

$\xrightarrow{N\to\infty} 0$, because $|\alpha_N| + |\beta_N| \xrightarrow{N\to\infty} 0$.

· $$var_{P_N^*}\left(Y_k^{(N)}\right) = E_{P_N^*}\left[\underline{\left(R_k^{(N)}\right)^2} + \left(R_k^{(N)}\right)^2\left(-R_k^{(N)} + \frac{1}{4}R_k^{(N)2}\right)\right]$$

$$-\left(E_{P_N^*}\left[\underline{R_k^{(N)}} - \frac{1}{2}R_k^{(N)2}\right]\right)^2$$

Only the underlined terms are relevant, because

$\cancel{(i).\ E_{P_N^*}\left[R_k^{(N)}\right]^2 \to \frac{\sigma^2}{cN}}$

(i) $$\left|E_{P_N^*}\left[R_k^{(N)i}\right]\right| \leq E_{P_N^*}\left[R_k^{(N)2}\right] \max\{|\alpha_N|^{i-2}, |$$

for $i \geq 3$

(ii) $\left| E_{P_N^*}\left[R_k^{(N)^2}\right] E_{P_N^*}\left[R^{(N)j}\right] \right|$

$$\leq E_{P_N^*}\left[R_k^{(N)^2}\right] \max\{|\alpha_N|^j, |\beta_N|^j\}$$

for $j = 1, 2$

and $\sum_{k=1}^{N} E_{P_N^*}\left[R_k^{(N)^2}\right]$ converges,

as we will see now.

$$\sum_{k=1}^{N} E_{P_N^*}\left[R_k^{(N)^2}\right] = \sum_{k=1}^{N}\left(\operatorname{var}(R_k^{(N)}) + r_N^2\right)$$

$\xrightarrow{N\to\infty} \sigma^2 T$, because $N r_N$ is a null sequence.

This also shows

$$\sum \operatorname{var}(Y_k^{(N)}) \xrightarrow{N\to\infty} \sigma^2 T.$$

Step 3: We show that

$$E_{P_N^*}\left[|\Delta_N|\right] \longrightarrow 0$$

$$E_{P_N^*}\left[|\Delta_N|\right] \leq \delta(\alpha_N, \beta_N) \sum_{k=1}^{N} E_{P_N^*}\left[R_k^{(N)2}\right]$$

$$\xrightarrow{N\to\infty} 0$$

Step 4: Prop. 114 $\Rightarrow$ $\sum^{N} Y_k^{(N)} \xrightarrow{w} N(rT - \frac{\sigma^2}{2}T, \sigma^2 T)$

We also know $E_{P_N^*}[|\Delta_N|] \xrightarrow{N\to\infty} 0$

Exercise: $\left(\sum Y_k^{(N)}\right) + \Delta_N \xrightarrow{w} N(rT - \frac{\sigma^2}{2}T, \sigma^2 T)$

(hint: Consider uniformly continuous, bounded functions first.

and use then a bump-function.)

$\square$

<u>Remark 115:</u> The theorem considers a market model for the time T. at the limit.
How do we get at the limit a market model which includes all times $t \in [0,T]$?

Take $t \in [0,T]$

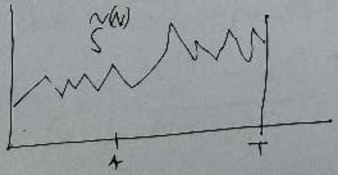
$$\tilde{S}^{(N)}_t = S^{(N)}_{\lfloor t \frac{N}{T} \rfloor} + \left(t\frac{N}{T} - \lfloor t \frac{N}{T} \rfloor \right) \cdot \left(S^{(N)}_{\lfloor t \frac{N}{T} \rfloor + 1} - S^{(N)}_{\lfloor t \frac{N}{T} \rfloor} \right)$$

Then $\tilde{S}^{(N)}_t \xrightarrow{w} S_t = S_0 \exp\left(\sigma W_t + \left(r - \frac{\sigma^2}{2}\right) t \right)$

Where $W_t \sim N(0,t)$

$$\tilde{X}^{(N)}_t = \frac{\tilde{S}^{(N)}_t}{(1+r_N)^{\lfloor t \frac{N}{T} \rfloor}}$$

$$\xrightarrow{w} X_t = S_0 \exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t \right)$$



Claim: Consider $(M,d) = (C[0,T], \|\cdot\|_\infty)$

Then $\tilde{X}^{(N)} \xrightarrow{w} X$

(these are random paths)

where $X_t := S_0 \exp(\sigma W_t - \frac{\sigma^2}{2} t)$, $t \in [0,T]$

such that $(W_t, t \in [0,T])$ is a Wiener process (or Brownian motion) w.r.t. some P^*, i.e.

- $t \longmapsto W_t$ is continuous P^*-a.s.
- $W_0 \equiv 0$
- $\forall\ 0 = t_0 < t_1 < t_2 < \dots < t_N = T$

$(W_{t_1} - W_0, W_{t_2} - W_{t_1}, W_{t_3} - W_{t_2}, \dots)$

are independent and

$W_{t_i} - W_{t_{i-1}} \sim N(0, t_i - t_{i-1})$

(Remark after 5.59)