INTRODUCTION TO MATHEMATICAL FINANCE
PROBLEM SHEET 13

PROF. DANIEL SKODLERACK

Problem 1 (10 points, Black-Scholes model, Δ-Γ-hedge)**.** In the Black-Scholes model find a position with a zero Δ and a zero Γ and vega $\mathcal{V} = 1$.

Problem 2 (20 points,stopping times)**.** Let $((\underline{S}_t)_{t=\overline{0,T}}, (\mathcal{F}_t)_{t=\overline{0,T}}$ in $\mathcal{F}, \mathrm{P})$ be a multi-period market model with a bond $\mathrm{B} = \mathrm{S}^{(0)}$ and a risky asset S.

(i) Let τ, σ be stopping times. Prove that $\tau \wedge \sigma$, $\tau \vee \sigma$ and $\tau + \sigma$ are stopping times.

(ii) Let τ_λ, $\lambda \in \Lambda$, be a family of stopping times. Prove that their essential supremum is a stopping time.

(iii) Consider the following map.

$$\tau := \inf\{t \geqslant 0 |\ \mathrm{S}_t = \sup_{0 \leqslant s \leqslant \mathrm{T}} \mathrm{S}_s\}$$

Is τ a stopping time? If yes then give a proof. Otherwise give a counter example.

(iv) Suppose now that the market model is continuous, i.e. the range for t is $[0, \infty)$, and suppose that all trajectories of S are continuous. Given a reel number c prove that the map

$$\tau = \inf\{t \geqslant 0 |\ \mathrm{S}_t \geqslant c\}$$

is a stopping time. (Recall that in continuous time a map $\tau : \Omega \to [0, \infty]$ is called a stopping time if for all non-negative real numbers t the set $\{\tau \leqslant t\}$ is an element of $\mathcal{F}_t$.

Problem 3 (10 points, Black-Scholes PDE)**.** Find all continuous contingent claims $f(\mathrm{S}_\mathrm{T})$ in the Black-Scholes model which are not time sensitive at any time to maturity and which satisfy

$$f(x) \leqslant (1 + x)^p,\ x > 0,$$

for one positive number p.

Problem 4 (20 points, optimal stopping times for American options)**.** Consider Problem 9.3 and let K be a positive number. Find an optimal exercise strategy for the American put with strike K for the given measure P and the martingale measure for the market model, i.e. for both cases choose an optimal stopping time and compute its value for every trajectory.

Date: Please hand in before the lecture by **19,.06.2022**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference.