INTRODUCTION TO MATHEMATICAL FINANCE
PROBLEM SHEET 12

PROF. DANIEL SKODLERACK

Problem 1 (10, sets whose boundary is not charged)**.** Let (M, d) be a metric space and μ be a finite positive measure on the Borel σ-algebra of (M, d). Let A be a closed set of (M, d). Show that, for $\varepsilon > 0$, there is an open set U containing A with $\mu(U \backslash A) \leqslant \varepsilon$ such that the boundary of U is not charged by μ.

Problem 2 (10, strong Markov property)**.** Consider the following game. One starts with 1000 RMB and proceeds the following routine:

1. You pay 1 RMB for this round.
2. One throws the dice.
3. If the dice shows a number greater than 2 then you get payed 2 RMB. And if it shows a number smaller than 3 than get 0 RMB.

This game continues until you run out of money. What is the probability that this game ends after a finite number of rounds?

Problem 3 (10, weak convergence with error term)**.** Let, for $N \in \mathbb{N}$, $(\Omega_N, \mathcal{F}_N, P_N)$ be a probability space with random variables X_N, Y_N such that, for N to infinity, X_N converges weakly to a probability measure μ and such that

$$\lim_{N\to\infty} E[|Y_N|] = 0.$$

Show that $X_N + Y_N$ converges weakly to μ.

Problem 4 (10, greeks)**.** We consider the Black-Scholes model for a stock. Recall: This model is the weak limit of a sequence of CRR-models as in Theorem 112.

(i) The derivative of the price with respect to the volatility σ is called *Vega* $\mathcal{V}$. Show that vega is always positive.

(ii) Further greeks:
- Δ is the derivative of the price w.r.t. S_0.
- Θ is the derivative of the price w.r.t. the time.
- ϱ is the derivative of the price w.r.t. the interest rate r.

(a) Prove that the price $v(S_0, T)$ of the call option satisfies

$$v(S_0, T) = S_0\Delta - \frac{\varrho}{T}.$$

(b) Prove

$$\Theta = \frac{\sigma}{2T}\mathcal{V} + \frac{r}{T}\varrho.$$

(iii) The greek Γ is the sensitivity of Δ w.r.t. the price of the underlying asset, i.e. the derivative of Δ w.r.t. S_0. Show that the price of the call option is convex in S_0.

Date: Please hand in before the lecture by 31^{st} **of May 2022**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference.