DIFFERENTIAL TOPOLOGY
PROBLEM SHEET 2

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Problem 1 (10, projective space)**.** We consider the n-dimensional reel projective space $\mathrm{P}^n\mathbb{R}$, defined as the set of 1-dimensional reel subspaces of $\mathbb{R}^{n+1}$ with the following map:

$$d(l_1, l_2) := \inf\{d_2(v_1, v_2)|\ v_1 \in l_1,\ v_2 \in l_2 \text{ such that } d_2(v_1, v_1) = d_2(v_2, v_2) = 1\},$$

where d_2 is the Euclidean metric on $\mathbb{R}^{n+1}$.

(i) Show that d is a metric.

(ii) Find a C^1-differential structure on $\mathrm{P}^n\mathbb{R}$ such that the map

$$\mathbb{R}^n \to \mathrm{P}^n\mathbb{R}$$

which maps $(x_1, \ldots, x_n)$ to the line through $(1, x_1, \ldots, x_n)$ is C^1-differentiable.

Problem 2 (10, constant maps)**.** Let $f : \mathrm{M} \to \mathrm{N}$ be a C^1-map between smooth manifolds. Suppose that M is connected and the derivative of f vanishes everywhere. Show that the image of f consists just of one element.

Problem 3 (10, derivations)**.** Prove Proposition 1.26 (b), more precisely find for every C^∞-manifold M and every point $\mathrm{P} \in \mathrm{M}$ an isomorphism

$$\Phi_{\mathrm{P,M}} : \mathrm{Der}_\mathrm{P}(\mathrm{M}) \to \mathrm{T}_\mathrm{P}\mathrm{M}$$

such that for every pair of C^∞-manifolds and every $f \in \mathrm{C}^\infty(\mathrm{M}, \mathrm{N})$ the diagramm

$$\begin{array}{ccc} \mathrm{Der}_\mathrm{P}(\mathrm{M}) & \xrightarrow{\Phi_{\mathrm{P,M}}} & \mathrm{T}_\mathrm{P}\mathrm{M} \\ \downarrow & & \downarrow \\ \mathrm{Der}_{f(\mathrm{P})}(\mathrm{N}) & \xrightarrow{\Phi_{\mathrm{P,N}}} & \mathrm{T}_{f(\mathrm{P})}\mathrm{N} \end{array}$$

commutes, where the right map is the derivative of f at P and the left is the push forward of derivations.

Problem 4 (20, torus)**.** Given the torus M in $\mathbb{R}^3$ as a submanifold, find a C^1-map f from M to itself such that the derivative of f in terms of coordinates (θ, α) (see Example 1.20) has the form

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

everywhere. Show that this map is not homotopic to the identity map of M. Compute the affine tangent planes of M at the points $\mathrm{P} := (-r_1 - r_2, 0, 0)$ and $f(\mathrm{P})$ as subsets of $\mathbb{R}^3$ and compute the derivative of f at P in terms of the affine tangent planes.

Date: Please hand in before the lecture by **24th of February 2023**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference.