

**COMMUTATIVE ALGEBRA
EXERCISE SHEET 12**

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Problem 1 (Map between Grothendieck spectra 10 points). Let $\varphi : R \rightarrow S$ be a ring homomorphism of unitary rings. Consider the map between the Grothendieck spectra:

$$\varphi^* : \text{Spec}(S) \rightarrow \text{Spec}(R)$$

given by $\varphi^*(\mathfrak{Q}) := \varphi^{-1}(\mathfrak{Q})$. Show that the image of φ^* is the set of those prime ideals \mathfrak{P} of R which satisfy

$$\mathfrak{P} = \varphi^{-1}(\varphi(\mathfrak{P})S).$$

Problem 2 (Normality is a local property. 20 points). Suppose that R is an integral domain. Show at first that R is equal to the intersection of all localizations with respect to the maximal ideals. Secondly, prove that the following assertions are equivalent:

- (i) R is normal.
- (ii) All localizations of R with respect to maximal ideals are normal.

Problem 3 (10 points). Show that a Dedekind domain with only finitely many prime ideals is a principal ideal domain.

Date: Please hand in before the lecture on Tuesday **14.12.2021**. For all exercises the results need to be proven. You are allowed to use results from the Abstract Algebra course.