COMMUTATIVE ALGEBRA
EXERCISE SHEET 4

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Problem 1 (10 points, radical of an ideal)**.** Let R be a ring.

(i) Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R. Show:

$$\sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}} = \sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{\mathfrak{a}\mathfrak{b}}.$$

(ii) Let $\{\mathfrak{a}_i, i \in \mathrm{I}\}$, be a $\subseteq$-totally ordered non-empty set of radical ideals. Show that $\bigcup_{i\in\mathrm{I}} \mathfrak{a}_i$ is a radical ideal of R.

Problem 2 (10 points, number rings)**.** We consider the ring $\mathrm{R} := \mathbb{Z}[\sqrt{-5}]$. Show $\mathrm{V}_{min}((6)_{\mathrm{R}})$ consists of exactly three elements, namely the ideals

$$(2, 1+\sqrt{-5})_{\mathrm{R}},\ (3, 1+\sqrt{-5})_{\mathrm{R}},\ (3, 1-\sqrt{-5})_{\mathrm{R}}.$$

Problem 3 (10 points, idempotents)**.** Find all idempotents of the ring

(i) $\mathbb{Z}[X]/(\mathrm{P})_{\mathbb{Z}[X]},\ \mathrm{P}(X) = X^4 + X^3 - 7X^2 - X + 6.$

(ii) $\mathbb{Z}/p^n\mathbb{Z},\ p$ a prime and $n \in \mathbb{N}$.

Problem 4 (10 points, minimal prime ideals)**.** Let $\mathfrak{a}$ be the ideal of $\mathrm{R}[X_i | i \in \mathbb{N}]$ generated by all monomials $X_iX_j, i \neq j$. Compute $\mathrm{V}_{min}(\mathfrak{a})$.

Date: Please hand in before the lecture by **12.10.2021**. For all exercises the results need to be proven. You are allowed to use results from the Abstract Algebra course.