COMMUTATIVE ALGEBRA
EXERCISE SHEET 2

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Problem 1 (10 points)**.** Let M be an R-module and N be a submodule of M. Show that there is unique R-module structure on M/N such that the projection

$$\pi : \mathrm{M} \to \mathrm{M}/\mathrm{N},\ \pi(m) := [m]_{\mathrm{N}},$$

is an R-module homomorphism.

Problem 2 (5+5 points)**.** Let M and N be two R-submodules of an R-module P. The set

$$(\mathrm{M} : \mathrm{N}) := \{x \in \mathrm{R} |\ x\mathrm{N} \subseteq \mathrm{M}\}$$

is called *the annihilator of* N *relative to* M. If M is the zero-module then we denote (0 : N) by ann(N) and call it the *annihilator of* N.

(i) Show that (M : N) is an ideal of R.
(ii) Show (M : N) = ann((M + N)/M).
(iii) Show that for every element a of R the sequence

$$\mathrm{R}/(\mathrm{I} : (a)_{\mathrm{R}}) \to \mathrm{R}/\mathrm{I} \to \mathrm{R}/(\mathrm{I} + (a)_{\mathrm{R}})$$

is exact.

(If P is the ring R then (M : N) is also called the ideal quotient of M by N.)

Problem 3 (5+5+5 points (ex. non-commutative ring B_n))**.** Let B_n be the set of upper triangular $n \times n$-matrices ($n \geqslant 2$) with entries in a field R. Then $\mathrm{R}^{n\times 1}$ is a B_n-left module via matrix multiplication. Show that the set of B_n-submodules of $\mathrm{R}^{n\times 1}$ is totally ordered with respect to the inclusion order.

Problem 4 (10 points)**.** Let M be an R-module and S be a subset of M. Show that the submodule of M generated by S is equal to the set

$$\{\Sigma_{i=1}^{l} r_i s_i \mid\ l \in \mathbb{N}, r_i \in \mathrm{R}, s_i \in S\}.$$

Date: Please hand in before the lecture by **28.09.2021**. For all exercises the results need to be proven. You are allowed to use results from the Abstract Algebra course.