COMMUTATIVE ALGEBRA
EXERCISE SHEET 1

PROF. DANIEL SKODLERACK

Problem 1 (5+5 points)**.** (i) Let $\varphi: R \to S$ be a homomorphism of unitary rings (This includes $\varphi(1_R) = 1_S$). Show that S is an R-module via

$$*: R \times S \to S,\ r * s := \varphi(r)s$$

on the group $(S, +)$.

(ii) Let M be an R-module. Show for all $r \in R, m \in M$:

$$(-r)m = -(rm) = r(-m),\ 0_R m = r0_M = 0_M.$$

Problem 2 (5+5 points)**.** (i) Show that $M := \mathbb{R}^\times$ with the abelian group structure

$$x \odot y := xy$$

(multiplication in $\mathbb{R}$) is a $\mathbb{Z}$-module via

$$z * x := x^z,$$

(ii) Show that there is no $\mathbb{Q}$-vector space structure $\tilde{*}$ on M such that the restriction $\tilde{*}|_{\mathbb{Z}\times M}$ is equal to $*$.

Problem 3 (5+5+5 points)**.** (i) Let $V \subseteq \mathbb{R}^2$ be the zero set of the polynomial $P := Y^n - X^m \in \mathbb{R}[X, Y]$ with given fixed coprime positive integers n, m. Let $Q(X, Y)$ be another polynomial over $\mathbb{R}$ vanishing on all points of V. Show that P divides Q in $\mathbb{R}[X, Y]$.

(ii) Show that the coordinate ring $\mathbb{R}[V] = \mathbb{R}[X, Y]/(P)$ is $\mathbb{R}$-algebra isomorphic to $\mathbb{R}[T^n, T^m]$.

(iii) Prove that the polynomial P is irreducible over $\mathbb{R}$.

Problem 4 (10 points)**.** Let n be a positive integer. We introduced a graph of homothety classes of lattices of $W := \mathbb{Q}^n$ in the lecture. Prove that this graph is connected.

Date: Please hand in before the lecture by **21.09.2021**. For all exercises the results need to be proven. You are allowed to use results from the Abstract Algebra course.