Hint for Problem 11.1.:

Let $E|F$ be a field extension and $\alpha_1, \ldots, \alpha_\ell \in E$. One has to show

for $\ell > 1$:

If $F[\alpha_1, \ldots, \alpha_\ell] = F(\alpha_1, \ldots, \alpha_\ell)$

then $F[\alpha_1, \ldots, \alpha_{\ell-1}] = F(\alpha_1, \ldots, \alpha_{\ell-1})$

[i.e. if $F[\alpha_1, \ldots, \alpha_\ell]$ is a field then $F[\alpha_1, \ldots, \alpha_{\ell-1}]$ is a field.]

Denote $R_2 := F[\alpha_1, \ldots, \alpha_\ell]$ and

$R_1 := F[\alpha_1, \ldots, \alpha_{\ell-1}]$

So, suppose $Q(R_2) = R_2$. (field of fractions)

α_ℓ is algebraic over $Q(R_1)$, because

$F[\alpha_1, \ldots, \alpha_\ell] = F(\alpha_1, \ldots, \alpha_\ell) = Q(R_1)[\alpha_\ell]$.

$\Rightarrow \exists \beta \in R_1 \setminus \{0\}$ $\exists$ $P \in R_1[X] \setminus R_1$ monic s.t. $P(\beta\alpha_\ell) = 0$. $(*)$

Write $\gamma := \beta\alpha_\ell$.

(We say that γ is <u>integral over R_1</u>)

<u>Claim 1</u>: Every element of R_2 is integral over R_1.

<u>Proof</u>: $R_2 = R_1[\alpha_\ell] \underset{\substack{\uparrow \\ \beta \in R_1 \setminus \{0\}}}{=} R_1[\beta\alpha_\ell]$

$= R_1[\gamma]$.

Consider the polynomial P in $(*)$

$d := \deg P$.

$\Rightarrow \quad R_2 = R_1 + R_1\gamma + \dots + R_1\gamma^{d-1}$.

Take $\delta \in R_2$

To show: δ is integral over R_1, 3

i.e. $\exists Q \in R_1[X] \setminus R_1$ monic: $Q(\delta) = 0$.

We have $\delta \gamma^i \in R_2$, $i = 0, \dots, d-1$.

$\Rightarrow \exists\; A \in M_d(R_1)$:

$$\begin{pmatrix} \delta \\ \delta\gamma \\ \delta\gamma^2 \\ \vdots \\ \delta\gamma^{d-1} \end{pmatrix} = A \begin{pmatrix} 1 \\ \gamma \\ \gamma^2 \\ \vdots \\ \gamma^{d-1} \end{pmatrix}$$

$$\Rightarrow \quad (A - \delta I_d) \begin{pmatrix} 1 \\ \gamma \\ \gamma^2 \\ \vdots \\ \gamma^{d-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\Rightarrow \det(A - \delta I_n) \begin{pmatrix} 1 \\ \gamma \\ \vdots \\ \gamma^{d-1} \end{pmatrix}$$

$$= \operatorname{Adj}(A - \delta I_d)(A - \delta I_d) \begin{pmatrix} 1 \\ \gamma \\ \vdots \\ \gamma^{d-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\Rightarrow \quad \det(A - \delta I_n) = 0$ (Noether's determinant trick)

Take $Q(X) := (-1)^d \det(A - X I_n)$

$\square$ (Claim 1)

<u>Claim 2:</u> R_1 is a field.

<u>Proof:</u> Take $\alpha \in R_1 \setminus \{0\}$.

$\Rightarrow \frac{1}{\alpha} \in R_2$ (because R_2 is a field.)

$\frac{1}{\alpha}$ is integral over R_1

$\Rightarrow \exists_{d>0} \exists a_0, \ldots, a_{d-1} \in R_1$

$$\left(\frac{1}{\alpha}\right)^d + a_{d-1}\left(\frac{1}{\alpha}\right)^{d-1} + \ldots + a_1\left(\frac{1}{\alpha}\right) + a_0 = 0$$

$$\Rightarrow \frac{1}{\alpha} + \underbrace{a_{d-1} + a_{d-2}\alpha + \ldots + a_1\alpha^{d-2} + a_0\alpha^{d-1}}_{\in R_1} = 0$$

$\Rightarrow \frac{1}{\alpha} \in R_1$ $\square$