ABSTRACT ALGEBRA
EXERCISE SHEET 13

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Problem 1 (10 points)**.** Let F be a field. Let p be its characteristic. We call F *perfect* if every algebraic extension of F is separable. Show that the following assertions are equivalent.

(i) The field F is perfect.

(ii) If p is positive then the Frobenius homomorphism of F

$$x \in \mathrm{F} \mapsto x^p \in \mathrm{F}$$

is surjective.

Problem 2 (10 points)**.** (i) Let E|F be a finite field extension and E_1 be an intermediate field, i.e.

$$\mathrm{F} \subseteq \mathrm{E}_1 \subseteq \mathrm{E}.$$

Let α be an element of E which is separable over F. Show that α is separable over E_1.

(ii) Let $\mathrm{E}|\mathrm{E}_2|\mathrm{E}_1|\mathrm{F}$ be a algebraic field extensions and suppose that $\mathrm{E}_2|\mathrm{E}_1$ is finite. Let ϕ be a field homomorphism from E_1 into an algebraic closure $\bar{\mathrm{F}}$ of F. Prove that the following assertions are equivalent.

(a) $\mathrm{E}_2|\mathrm{E}_1$ is separable.

(b) There are exactly $[\mathrm{E}_2 : \mathrm{E}_1]$ extensions of ϕ to E_2.

Problem 3 (10 points)**.** Let E|F be an algebraic field extension and $\mathrm{E}_1|\mathrm{F}$ and $\mathrm{E}_2|\mathrm{F}$ be separable subextensions. Prove that the compositum $\mathrm{E}_1\mathrm{E}_2$ is separable over F,

Problem 4 (10 points)**.** Let p be an odd prime number and F be a finite field of characteristic p. Find the largest field E_1 in between $\mathrm{F}(X)$ and $\mathrm{F}(X^{\frac{1}{2p}})$ such that $\mathrm{E}_1|\mathrm{F}(X)$ is separable.

Date: Please hand in before the lecture by **27.05.2021**. For all exercises the results need to be proven.