ABSTRACT ALGEBRA
EXERCISE SHEET 12

PROF. DANIEL SKODLERACK

Problem 1 (10 points)**.** Let $\mathrm{E}_1|\mathrm{F}$ and $\mathrm{E}_2|\mathrm{F}$ be two finite extensions inside a field extension $\mathrm{E}|\mathrm{F}$. Show that

$$[\mathrm{E}_1\mathrm{E}_2 : \mathrm{F}] \leqslant [\mathrm{E}_1 : \mathrm{F}][\mathrm{E}_2 : \mathrm{F}].$$

Problem 2 (10 points)**.**

(i) Let A_i, $i \in \mathbb{N}$, be non-empty sets such that $\mathrm{A}_i \supseteq \mathrm{A}_{i+1}$ for all $i \in \mathbb{N}$. Show that $(f_{i,j})_{i\leqslant j\in\mathbb{N}}$ is a projective system, where $f_{i,j}$ is the inclusion of A_j into A_i, and show that there is a bijection from $\varprojlim_{\mathbb{N}}\mathrm{A}_i$ to $\bigcap_{i\in\mathbb{N}}\mathrm{A}_i$, if the latter intersection is non-empty.

(ii) Consider the projective limit

$$\varprojlim_{\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$$

for a prime number p. This ring is denoted by $\hat{\mathbb{Z}}_p$ and is called the ring of p-adic integers. Show that there is a solution of $x^2 = 2$ in $\hat{\mathbb{Z}}_7$.

Problem 3 (10 points)**.** Let $\mathrm{E} = \mathrm{F}(\alpha)$ be algebraic over F of odd degree. Show that $\mathrm{E} = \mathrm{F}[\alpha] = \mathrm{F}[\alpha^2]$.

Problem 4 (10 points)**.** Prove Proposition 122(ii).

Date: Please hand in before the lecture by **20.05.2021**. For all exercises the results need to be proven.