ABSTRACT ALGEBRA
EXERCISE SHEET 8

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Problem 1 (10+10* points)**.** Let M be a non-empty set. A non-empty subset F of $\mathfrak{P}(\mathrm{M})$ is called a *filter* of M if for every $\mathrm{N}_1, \mathrm{N}_2 \in \mathrm{F}$ we have

$$\emptyset \neq \mathrm{N}_1 \cap \mathrm{N}_2 \in \mathrm{F}.$$

A filter which is maximal with respect to inclusion is called an *ultrafilter.*

(i) Show that a given filter F of M is an ultrafilter if and only if for every set $\mathrm{N} \subseteq \mathrm{M}$ we have $\mathrm{N} \in \mathrm{F}$ or $\mathrm{M}\backslash\mathrm{N} \in \mathrm{F}$.

(ii) Let m_0 be an element of M. Show that the set

$$\{\mathrm{N} \subseteq \mathrm{M} | m_0 \in \mathrm{N}\}$$

is an ultrafilter. (Those ultrafilters are called *principal.*)

(iii) Suppose that M is infinite. Show that there exists an ultrafilter of M which is not principal.

(iv) Let $\mathfrak{m}$ be a maximal ideal of $\mathrm{Map}(\mathrm{M}, \mathbb{R})$. Show that

$$\{\mathrm{zeros}(f) |\ f \in \mathfrak{m}\}$$

is an ultrafilter of M. ($\mathrm{zeros}(f) = \{m \in \mathrm{M} |\ f(m) = 0\}$)

Problem 2 (10 points)**.** Let R be an integral domain. Prove Proposition 80 from the lecture.

Problem 3 (10 points)**.** Let W be an $\mathbb{R}$-vector space. We define on

$$\mathbb{R}_{\mathrm{W}} := \mathbb{R} \times \mathrm{W}$$

an addition and a multiplication via:

$$(r, w) + (s, v) := (r + s, w + v),\ (r, w) * (s, v) := (rs, sw + rv),$$

for $r, s \in \mathbb{R}$ and $w, v \in \mathrm{W}$.

(i) Show that $(\mathbb{R}_{\mathrm{W}}, +, *)$ is a non-zero commutative unitary ring.

(ii) Prove that $\mathbb{R}_{\mathrm{W}}$ is noetherian if and only if W is finite dimensional as an $\mathbb{R}$-vector space.

Problem 4 (10 points*)**.** Let R be a commutative non-zero unitary ring and $\mathfrak{a}$ be an ideal of R. Suppose $\mathfrak{p}_i,\ i = 1, \ldots, l$, are prime ideals of R such that

$$\mathfrak{a} \subseteq \bigcup_{i=1}^{l} \mathfrak{p}_i.$$

Prove that there exists an index $i \in \mathbb{N}^{\leq l}$ such that $\mathfrak{a} \subseteq \mathfrak{p}_i$.

Date: Please hand in before the lecture by **20.04.2021**. For all exercises the results need to be proven.