ABSTRACT ALGEBRA
EXERCISE SHEET 6

PROF. DANIEL SKODLERACK

Problem 1 (10 points)**.** Find all maximal left ideals of $M_2(\mathbb{Z}/4\mathbb{Z})$.

Problem 2 (10+10* points)**.**

(i) Read the proof of "Axiom of choice implies Zorn's Lemma" in the book "Algebra" by Serge Lang and point out the exact position where the axiom of choice is applied.

(ii) Let G be a finitely generated group. Show that G contains a maximal proper subgroup, i.e. a subgroup N different from G such that no subgroup H of G different from G and N contains N.

(iii) Does (ii) still hold if we remove the condition of being finitely generated?

Problem 3 (10 points)**.** Let $\varphi : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be a ring homomorphism such that $\varphi(rI_2) = r(I_2)$ for all reel numbers r. Show that there is an invertible matrix A such that for all elements $X \in M_2(\mathbb{R})$ we have

$$\varphi(X) = AXA^{-1}.$$

Problem 4 (10 points)**.** We consider a non-empty set M and the ring

$$(\mathrm{Map}(M, \mathbb{R}), +, \cdot),$$

given by pointwise addition and multiplication:

$$(f_1 + f_2)(m) := f_1(m) + f_2(m),\ (f_1 \cdot f_2)(m) := f_1(m)f_2(m).$$

Find all maximal ideals.

Date: Please hand in before the lecture by **06.04.2021**. For all exercises the results need to be proven.