ABSTRACT ALGEBRA
EXERCISE SHEET 5

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Problem 1 (10 points, commutator subgroups)**.** A group G is called *perfect* if G coincides with its commutator subgroup.

(i) Compute $[\mathrm{SL}_n(\mathbb{R}), \mathrm{SL}_n(\mathbb{R})]$.

(ii) Show that $\mathrm{SL}_2(\mathbb{Z})$, the group of matrices with integer coefficients and determinant 1, is not perfect.

Problem 2 (10 points)**.** (i) Show that for a prime number p the triple $(\mathbb{Z}/p\mathbb{Z}, +, \cdot)$ with $[z_1]_p \cdot [z_2]_p := [z_1 z_2]_p$ is a commutative unitary ring in which every non-zero element has a multiplicative inverse.

(ii) Show that $(\mathbb{Z}/30\mathbb{Z}, +)$ is group isomorphic to $((\mathbb{Z}/31\mathbb{Z})^\times, \cdot)$.

(iii) Find all integer solutions of the equation

$$X^3 + Y^3 = 4 + Z^3.$$

Problem 3 (10 points)**.** (i) Let G be a group and H be a subgroup of G. Show that H contains a normal subgroup of G which contains all normal subgroups of G contained in H.

(ii) Let G be a group and R a subset of G. We call, for $g \in \mathrm{G}$, the set

$$\mathrm{C}(g) := \{g' g g'^{-1} |\ g' \in \mathrm{G}\}$$

the G-*conjugacy class of* g. Prove that the normal subgroup of G generated by R is equal to the subgroup of G generated by the union of all G-conjugacy classes of elements of R.

Problem 4 (10 points)**.** Show that the group

$$\mathrm{G} := \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} |\ z_1, z_2, z_3 \in \mathbb{Z} \right\}$$

with the usual matrix multiplication is isomorphic to

$$\langle a, b, c |\ ab = ba,\ bc = cb,\ aca^{-1}c^{-1} = b \rangle. \tag{0.1}$$

We say that G is *presented* by (0.1).

Date: Please hand in before the lecture by **30.03.2021**. For all exercises the results need to be proven.