ABSTRACT ALGEBRA
EXERCISE SHEET 4

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Problem 1 (10 points, Noether's first isomorphism theorem)**.** Let G be a group and H and N be normal subgroups of G such that H is contained in N. Show:

$$(\mathrm{G}/\mathrm{H})/(\mathrm{N}/\mathrm{H}) \cong \mathrm{G}/\mathrm{N},$$

as groups.

Problem 2 (5+(5+10*) points, graphs)**.** Let $\Gamma := (\mathrm{V}, \mathrm{E})$ be a graph, i.e. V is a non-empty set (vertexes) and E is a subset of

$$\{W \subseteq \mathrm{V} | \ |\mathrm{W}| = 2\}$$

(edges). A bijection f of V is called an automoprphism of Γ (we write $\mathrm{Aut}(\Gamma)$ for the set of those) if for all $x, y \in \mathrm{V}$ we have

$$\{x, y\} \in \mathrm{E} \text{ iff. } \{f(x), f(y)\} \in \mathrm{E}.$$

(i) Show that $\mathrm{Aut}(\Gamma)$ is a group.
(ii) Consider $\mathrm{V} = \{1, 2, 3, 4\}$.
  (a) Find $\mathrm{Aut}(\Gamma)$ for

$$\mathrm{E} = \{\{x, y\} | x, y \in \mathrm{V}, x \neq y\} \backslash \{\{1, 2\}\}.$$

  (b) Find all subgroups H of $\mathfrak{S}_4$ for which there is a set of edges E such that $\mathrm{Aut}(\Gamma) = \mathrm{H}$.

Problem 3 (10 points, dihedral groups)**.** We consider the graph $\Gamma = (\mathrm{V}, \mathrm{E})$ for $\mathrm{V} := \mathbb{Z}/n\mathbb{Z}$, $n \geqslant 3$, and

$$\mathrm{E} = \{\{[z], [z+1]\} \mid \ z \in \mathbb{Z}\}.$$

Compute the order of the automoprphism group of Γ.

Problem 4 (10 points, free groups)**.** Let F_n, $n \in \mathbb{N}$, be the free group on n letters. Show that its abelization $\mathrm{F}_n/[\mathrm{F}_n : \mathrm{F}_n]$ is group isomorphic to $(\mathbb{Z}^n, +)$.

Date: Please hand in before the lecture by **23.03.2021**. For all exercises the results need to be proven.