ABSTRACT ALGEBRA
EXERCISE SHEET 3

PROF. DANIEL SKODLERACK

Problem 1 (10 points)**.** Let n and m be positive integers. Find all group homomorphisms from $(\mathbb{Z}/n\mathbb{Z}, +)$ to $(\mathbb{Z}/m\mathbb{Z}, +)$.

Problem 2.

(i) Let $(\mathrm{G}_i, *_i)$, $i \in \mathrm{I}$, be a family of groups with identity $e_i \in \mathrm{G}_i$. Show that the following objects are groups:

(a) $\prod_{i\in\mathrm{I}} \mathrm{G}_i = \{(g_i)_{i\in\mathrm{I}} | \ g_i \in \mathrm{G}_i\}$ "direct product of groups"

(b) $\bigoplus_{i\in\mathrm{I}} \mathrm{G}_i = \{(g_i)_{i\in\mathrm{I}} | \ g_i \in \mathrm{G}_i, \ g_i = e_i \text{ a.a.}\}$ (a.a. means almost always, i.e. except of finitely many indexes) "direct sum of groups"

with the structure

$$(g_i)_{i\in\mathrm{I}} * (g_i')_{i\in\mathrm{I}} := (g_i *_i g_i')_{i\in\mathrm{I}}.$$

(ii) Find all normal subgroups of

$$(\mathbb{Z}/6\mathbb{Z}) \times \mathfrak{S}_3.$$

Problem 3 (10 points)**.** Consider the additive group $(\mathbb{Q}, +)$ with the subgroup $(\mathbb{Z}, +)$. Show that $\mathrm{G} := \mathbb{Q}/\mathbb{Z}$ is a torsion group, which has for every positive integer n exactly one subgroup H of order n, and further show that this subgroup is cyclic and

$$\mathrm{G}/\mathrm{H} \cong \mathrm{G}$$

as groups. (Recap: A group is called *torsion group* if every element has finite order.)

Problem 4 (10 points)**.** Let G be a group such that $(\mathrm{Aut}(\mathrm{G}), \circ)$ is cyclic. Show that G is abelian.

Problem 5 (10* points)**.** Find all subgroups of $\mathrm{GL}_2(\mathbb{R})$ of index 2.

Date: Please hand in before the lecture by **16.03.2021**. For all exercises the results need to be proven.