ABSTRACT ALGEBRA
EXERCISE SHEET 2

PROF. DANIEL SKODLERACK

Problem 1 (10 points)**.** Show that all groups of order less than 6 are abelian.

Problem 2 (10 points)**.**
(i) Show for $n \geqslant 2$ that the nth symmetric group $\mathfrak{S}_n$ is generated by the set

$$\{< 1, 2, \ldots, n >, < 1, 2 >\}.$$

(ii) Let n be a prime number and let $\tau \in \mathfrak{S}_n$ be a transposition. Show that the set

$$\{< 1, 2, \ldots, n >, \tau\}.$$

is a generating set for $\mathfrak{S}_n$.

Problem 3 (10 points)**.** Let n be a positive integer greater than 1.

(i) Finish the proof of Proposition 25, i.e. show that for all $\tau \in \mathfrak{S}_n$ we have $l(\tau) = \mathrm{inv}(\tau)$.
(ii) Write the permutations $< 1, n >$ and

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix},$$

the latter for $n = 5$, as a shortest product of transpositions of the form $< i, i+1 >$, $1 \leqslant i \leqslant n-1$.

Problem 4 (10 points)**.** Show that the sign map

$$\mathrm{sign} : \mathfrak{S}_n \to \{\pm 1\}$$

defined in the lecture is well-defined. More precisely: we defined, for $\tau \in \mathfrak{S}_n$, $\mathrm{sign}(\tau) = (-1)^k$ if τ can be written as a product of k transpositions. Show that the parity of k does not depend on the choice.

Problem 5 (10* points)**.** Let n be an integer greater than 1.

(i) Show that the set

$$\mathrm{G} := \{A \in \mathrm{M}_n(\mathbb{Z}) | \ \det(A) \in \{\pm 1\}\}$$

together with the matrix multiplication forms a group.
(ii) Find a generating subset of G of cardinality 3, i.e. a subset S of G which satisfies $\langle S \rangle_{\mathrm{G}} = \mathrm{G}$ and $|S| = 3$.

Date: Please hand in before the lecture by **09.03.2021**. For all exercises the results need to be proven.