ABSTRACT ALGEBRA
EXERCISE SHEET 1

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Problem 1 (10 points)**.** (i) Let M be a set. Show that $(\mathfrak{P}(M), \Delta)$ (the power set with the symmetric difference) is an abelian group.

(ii) Show that $(\mathbb{R} \times \mathbb{R}, *)$ with $*$ defined as

$$(a, b) * (c, d) := (a, bd)$$

is a non-abelian semigroup. Find all right- and left-units and find all elements which have a left-inverse with respect to the right-unit $(1, 1)$, i.e. all elements x of M for which there is an element y in M such that $y * x = (1, 1)$.

Problem 2 (10 points)**.** Find an example of an infinite monoid of whom all elements a satisfy: $a^2 = a$.

Problem 3 (10 points)**.** Let $(G, *)$ be a semigroup such that all translations

$$r_a, l_a : G \to G, \ a \in G,$$

defined via $l_a(b) := a * b$ and $r_a(b) := b * a$ are bijections. Show that $(G, *)$ is a group.

Problem 4 (10 points)**.** Suppose that $(M, *)$ is an abelian semigroup. Show that for all $a_1, \ldots, a_m$ and all $f \in \mathrm{Bij}(\{1, \ldots, m\})$ we have

$$a_1 * a_2 * \ldots * a_m = a_{f(1)} * a_{f(2)} * \ldots * a_{f(m)}.$$

Recall: $a_1 * a_2 * \ldots * a_m$ is iductively defined via

$$a_1 * a_2 * \ldots * a_m := (a_1 * a_2 * \ldots * a_{m-1}) * a_m, \ m > 1.$$

Problem 5 (10* points)**.** Let $(M, *)$ be a semigroup and let t be an element of M. We define a new structure

$$\odot : M \to M$$

via $a \odot b := a * t * b$. Show that

(i) $(M, \odot)$ is a monoid if and only if $(M, *)$ is a monoid in which t has an inverse.

(ii) $(M, \odot)$ is a group if and only if $(M, *)$ is a group.

Date: Please hand in before the lecture by **02.03.2021**. For all exercises the results need to be proven.