CUSPIDAL IRREDUCIBLE COMPLEX OR l-MODULAR REPRESENTATIONS OF QUATERNIONIC FORMS OF p-ADIC CLASSICAL GROUPS FOR ODD p

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Abstract. Given a quaternionic form G of a p-adic classical group (p odd) we classify all cuspidal irreducible representations of G with coefficients in an algebraically closed field of characteristic different from p. We prove two theorems: At first: Every irreducible cuspidal representation of G is induced from a cuspidal type, i.e. from a certain irreducible representation of a compact open subgroup of G, constructed from a β-extension and a cuspidal representation of a finite group. Secondly we show that two intertwining cuspidal types of G are up to equivalence conjugate under some element of G. [11E57][11E95][20G05][22E50]

1. Introduction

This work is the third part in a series of three papers, the first two being [35] and [37]. Let F be a non-Archimedean local field with odd residue characteristic p. The construction and classification of cuspidal irreducible representation complex or l-modular of the set of rational points $\mathbb{G}(\mathrm{F})$ of a reductive group $\mathbb{G}$ defined over F has already been successfully studied for general linear groups (complex case: [11] Bushnell–Kutzko, [31], [4], [32] Broussous–Secherre–Stevens; modular case: [44] Vigneras, [23] Minguez–Sechérre) and for p-adic classical groups ([42] Stevens, [21] Kurinczuk–Stevens, [20] Kurinczuk–Stevens joint with the author). In this paper we are generalizing from p-adic classical groups to their quaternionic forms. Let us mention [46] Yu, [15], [16] Fintzen and [19] Kim for results over reductive p-adic groups in general.

We need to introduce notation to describe the result. We fix a skew-field D of index 2 over F together with an anti-involution $(\bar{\ })$ on D and an ϵ-hermitian form

$$h : \mathrm{V} \times \mathrm{V} \to \mathrm{D}$$

on a finite dimensional D-vector space V. Let G be the group of isometries of h. Then G is the set of rational points of the connected reductive group $\mathbb{G}$ defined by h and $\mathrm{Nrd} = 1$, see §2.1. Let $\mathbf{C}$ be an algebraically closed field of characteristic $p_{\mathbf{C}}$ different from p. We only consider representations with coefficients in $\mathbf{C}$. At first we describe the construction of the cuspidal types (imitating the Bushnell–Kutzko–Stevens framework): A cuspidal type is a certain irreducible representation λ of a certain compact open subgroup J of G. The arithmetic core of λ is given by a skew-semisimple stratum $\Delta = [\Lambda, n, 0, \beta]$. It provides the following data (see [37] for more information):

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- An element β of the Lie algebra of G which generates over F a product E of fields in $A := \mathrm{End}_{\mathrm{D}}\,\mathrm{V}$. We denote the centralizer of β in G by G_β.
- A self-dual $\mathfrak{o}_{\mathrm{E}}$-$\mathfrak{o}_{\mathrm{D}}$-lattice sequence Λ of V which can be interpreted as a point of the Bruhat-Tits building $\mathfrak{B}(\mathrm{G})$ and as the image of a point Λ_β of the Bruhat-Tits building $\mathfrak{B}(\mathrm{G}_\beta)$ under a canonical map (see [33])
$$j_\beta : \mathfrak{B}(\mathrm{G}_\beta) \to \mathfrak{B}(\mathrm{G}).$$
.
- An integer $n > 0$ which is related to the depth of the stratum.
- Compact open subgroups of G: $\mathrm{H}^1(\beta,\Lambda) \subseteq \mathrm{J}^1(\beta,\Lambda) \subseteq \mathrm{J}(\beta,\Lambda)$.
- A set $\mathrm{C}(\Delta)$ of characters of $\mathrm{H}^1(\beta,\Lambda)$. (so-called self-dual semisimple characters)

The representation λ consists of two parts:

Part 1 is the arithmetic part: One chooses a self-dual semisimple character $\theta \in \mathrm{C}(\Delta)$, which admits a Heisenberg representation η on $\mathrm{J}^1(\beta,\Lambda)$ (see [10, §8] for these extensions) and then constructs a certain extension κ of η to $\mathrm{J}(\beta,\Lambda)$. ($\kappa$ having the same degree as η) Not every extension is allowed for κ. For example if Λ_β corresponds to a vertex in $\mathfrak{B}(\mathrm{G}_\beta)$ (which is the case for cuspidal types) we impose that the restriction of κ to a pro-p-Sylow subgroup of J is intertwined by G_β.

Part 2 is a representation of a finite group (This is the so called level zero part). Let $\mathrm{k_F}$ be the residue field of F. The group $\mathrm{J}(\beta,\Lambda)/\mathrm{J}^1(\beta,\Lambda)$ is the set of $\mathrm{k_F}$-rational point of a reductive group, here denoted by $\mathbb{P}(\Lambda_\beta)$. It is also the reductive quotient of the stabilizer $\mathrm{P}(\Lambda_\beta)$ of Λ_β in G_β. The pre-image $\mathrm{P}^0(\Lambda_\beta)$ of $\mathbb{P}(\Lambda_\beta)^0(\mathrm{k_F})$ (connected component) in $\mathrm{P}(\Lambda_\beta)$ is the parahoric subgroup of G_β corresponding to Λ_β. We choose an irreducible representation ρ of $\mathbb{P}(\Lambda_\beta)(\mathrm{k_F})$ whose restriction to $\mathbb{P}(\Lambda_\beta)^0(\mathrm{k_F})$ is a direct sum of cuspidal irreducible representations, and we inflate ρ to J, still called ρ, and define $\lambda := \kappa \otimes \rho$. Then λ is called a cuspidal type if $\mathrm{P}^0(\Lambda_\beta)$ is a maximal parahoric subgroup in G_β. (see. §7)

Then, we obtain the following classification theorem:

Theorem 1.1 (Main Theorem).
(i) Every irreducible cuspidal **C**-representation of G is induced by a cuspidal type. (Theorem 9.1)
(ii) If (λ,J) is a cuspidal type, then $\mathrm{ind}_{\mathrm{J}}^{\mathrm{G}}\,\lambda$ is irreducible cuspidal. (Theorem 7.3)
(iii) Two intertwining cuspidal types (λ,J) and (λ',J') are conjugate in G if and only if they intertwine in G. (Theorem 9.2)

The proof of the theorem needs several steps. We need a quadratic unramified field extension L|F and $\mathrm{G_L} := \mathrm{G}\otimes\mathrm{L}$ with its Bruhat-Tits building $\mathfrak{B}(\mathrm{G_L})$, and further the building $\mathfrak{B}(\tilde{\mathrm{G}}_{\mathrm{F}})$ of the general linear group $\tilde{\mathrm{G}}_{\mathrm{F}} = \mathrm{Aut}_{\mathrm{F}}(\mathrm{V})$.

(i) At first we show that every irreducible representation of G contains a self-dual semisimple character. (This is the most difficult part of the theory.), see Theorem 3.1. Mainly we use the canonical embeddings
$$\mathfrak{B}(\mathrm{G}) \hookrightarrow \mathfrak{B}(\mathrm{G_L}) \hookrightarrow \mathfrak{B}(\tilde{\mathrm{G}}_{\mathrm{F}})$$

together with unramified, here $\mathrm{Gal}(\mathrm{L}|\mathrm{F})$, and ramified, here $\mathrm{Gal}(\mathrm{F}[\varpi_{\mathrm{D}}]|\mathrm{F})$, Galois restriction to results of [41] and [40]. (cf. [14, §8.9])

(ii) We generalize the construction of β-extensions κ from the p-adic classical to the quaternionic case, see §6.

(iii) We prove that a self-dual semisimple character contained in a cuspidal irreducible representation needs to be skew, see §10.2.

(iv) We follow the proof in [42] to show the exhaustion part of Theorem 1.1, see §10.3. Here we needed to generalize the notion of subordinate decompositions, see §8.

(v) For the intertwining implies conjugacy part of Theorem 1.1 we use [37] and [21] and follow [20, §11], see §11.

In the appendix, see §A, we have added an erratum on a proposition in [40] which was used to show in *loc.cit.*, for p-adic classical groups G', that the coset of any non-G'-split fundamental stratum is contained in the coset of a skew-semisimple stratum. This was necessary because main statements in *loc.cit.* are used in the proof of existence of semisimple characters in irreducible representations. The proofs in the erratum were written by S. Stevens, the author of [42], in 2012, but not published yet.

In Appendix C we prove a Lemma which is very important for the exhaustion part (iv). It roughly says, that if in the search of a type in an irreducible representation of G with maximal parahoric, one has landed at a vertex (in the weak simplicial structure of $\mathrm{B}(\mathrm{G}_\beta)$) which does not support a maximal prahoric subgroup of G_β, then one can move along an edge to resume the search. This idea is disguised in [42, §7] and [24, Appendix], so that we found that it is worth to give a proof of this lemma, see Lemma C.1, in lattice theoretic terms.

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2. Notation

2.1. **Semisimple characters.** This article is a continuation of [37] and [35] which we call **I** and **II**. We mainly follow their notation, but there is a major change, see the remark below, and there are slight changes to adapt the notation to [42]. Let F be a non-Archimedean local field of odd residue characteristic p with valuation $\nu_{\mathrm{F}} : \mathrm{F} \to \mathbb{Z}$, valuation ring $\mathfrak{o}_{\mathrm{F}}$, valuation ideal $\mathfrak{p}_{\mathrm{F}}$, residue field k_{F} and we fix a uniformizer ϖ_{F} of F.

We fix an additive character ψ_{F} of F of level 1. We consider a quaternionic form G of a p-adic classical group as in **II**, i.e. $\mathrm{G} = \mathrm{U}(h)$ for an ϵ-hermitian form

$$h : \mathrm{V} \times \mathrm{V} \to (\mathrm{D}, \bar{(\ )}),$$

where D is a skew-field of index 2 and central over F together with an anti-involution $\bar{(\ )}: \mathrm{D} \to \mathrm{D}$ of D. The form h defines via its adjoint anti-involution an algebraic group $\mathbb{U}(h)$ defined over F. We denote with $\mathbb{G}$ the unital component of $\mathbb{U}(h)$, given by the additional equation Nrd = 1. By **II**.2.9 (see [25, 1.III.1]) the sets $\mathbb{G}(\mathrm{F})$ and $\mathbb{U}(h)(\mathrm{F})$ coincide with G, and we will consider $\mathbb{G}$ as the algebraic group associated to G. The ambient general linear group for G: $\mathrm{Aut}_{\mathrm{D}}(\mathrm{V})$, is denoted by $\tilde{\mathrm{G}}$. Let us recall that a stratum has the standard notation $\Delta = [\Lambda, n, r, \beta]$, i.e. the entries for Δ' are Λ', n', r', β' and for Δ_i are Λ^i, n_i, r_i and β_i. A semisimple stratum has a unique coarsest decomposition as a direct sum of simple strata: $\Delta = \oplus_{i\in I}\Delta_i$, in particular it decomposes $\mathrm{E} = \mathrm{F}[\beta]$ into a product of fields $\mathrm{E}_i = \mathrm{F}[\beta_i]$, provides idempotents via $1 = \sum_i 1^i$ and further decompositions

$$\mathrm{V} = \oplus_{i\in \mathrm{I}}\mathrm{V}^i,\ \mathrm{A} = \mathrm{End}_{\mathrm{D}}(\mathrm{V}) = \oplus_{i,j\in \mathrm{I}}\mathrm{A}^{ij},\ \mathrm{A}^{ij} := \mathrm{Hom}_{\mathrm{D}}(\mathrm{V}^i, \mathrm{V}^j).$$

The form h comes along with an adjoint anti-involution σ_h on A and an adjoint involution σ on $\tilde{\mathrm{G}}$, defined via $\sigma(g) := \sigma_h(g)^{-1}$. We denote by $\mathrm{C}_?(!)$ the centralizer of ! in ?, $\mathrm{B} := \mathrm{C}_{\mathrm{A}}(\beta)$ decomposes into $\mathrm{B} = \oplus_i \mathrm{B}^i$, $\mathrm{B}^i = \mathrm{C}_{\mathrm{A}^{ii}}(\beta_i)$. We write $\tilde{\mathrm{G}}_i$ for $(\mathrm{A}^{ii})^\times$. The adjoint anti-involution σ_h of h induces a map on the set of strata $\Delta \mapsto \Delta^\#$. Δ is called self-dual if Δ and $\Delta^\#$ coincide up to a translation of Λ, i.e. $n = n^\#, r = r^\#, \beta = \beta^\#$, and there is an integer k such that $\Lambda - k$, which is $(\Lambda_{j+k})_{j\in\mathbb{Z}}$, is equal to $\Lambda^\#$, i.e. Λ is *self-dual.* A self-dual o_{D}-lattice sequence is called *standard self-dual* if the o_{D}-period $e(\Lambda|o_{\mathrm{D}})$ is even and $\Lambda^\#(z) = \Lambda(1 - z)$ for all integer z. Further, in the self-dual semisimple case, the anti-involution σ_h induces an action of $\langle\sigma_h\rangle$ on the index set I of the stratum, and decomposes it as $\mathrm{I} = \mathrm{I}_+ \cup \mathrm{I}_0 \cup \mathrm{I}_-$, with the fixed point set I_0 and a section I_+ through all the orbits of length 2. We write $\mathrm{I}_{0,+}$ for $\mathrm{I}_0 \cup \mathrm{I}_+$. To a semisimple stratum Δ is attached a compact open subgroup $\tilde{\mathrm{H}}(\Delta)$ of $\tilde{\mathrm{G}}$ and a finite set of complex characters $\tilde{\mathrm{C}}(\Delta)$ defined on $\tilde{\mathrm{H}}(\Delta)$. If Δ is self-dual semisimple we define $\mathrm{C}(\Delta)$ as the set of the restriction of the elements of $\tilde{\mathrm{C}}(\Delta)$ to $\mathrm{H}(\Delta) := \tilde{\mathrm{H}}(\Delta) \cap \mathrm{G}$. Given a stratum Δ we denote by $\Delta(j-)$ the stratum $[\Lambda, n, r - j, \beta]$, if $n \geq r - j \geq 0$, for $j \in \mathbb{Z}$ and analogously we have $\Delta(j+)$. There is a major change of notation to *loc.cit.*:

Remark 2.1. We make the following convention for the notation. (Caution this is then different form the notation in *loc.cit.*.) Every object which corresponds to the general linear group $\tilde{\mathrm{G}}$ is going to get a $\tilde{(\ )}$ on top. Instead of $\mathrm{C}_-(\Delta)$ in **II** we write $\mathrm{C}(\Delta)$, and instead of $\mathrm{C}(\Delta)$ in **I** we write $\tilde{\mathrm{C}}(\Delta)$. Analogously for the groups and characters etc..

2.2. **Coefficients for the smooth representations.** In this paper we only consider smooth representations of locally compact groups H on **C**-vector spaces, where **C** is an algebraically closed field whose characteristic, denoted by $p_{\mathbf{C}}$, is different from p. We write $\mathcal{R}_{\mathbf{C}}(\mathrm{H})$ or $\mathcal{R}(\mathrm{H})$ for the category of those representations.

The theory of semisimple characters in **I**, **II** and [41], see also §2.1, is still valid for **C**, because **C** contains a full set $\mu_{p^\infty}(\mathbf{C})$ of p-power roots of unity. We fix a group isomorphism ϕ from $\mu_{p^\infty}(\mathbb{C})$ to $\mu_{p^\infty}(\mathbf{C})$ and define

$$\tilde{\mathrm{C}}^{\mathbf{C}}(\Delta) := \{\phi \circ \theta |\ \theta \in \tilde{\mathrm{C}}(\Delta)\},$$

and analogously $\mathrm{C}^{\mathbf{C}}(\Delta)$ for strata equivalent to self-dual semisimple strata and with respect to $\psi_{\mathrm{F}}^{\mathbf{C}} = \phi \circ \psi_{\mathrm{F}}$. We identify $\tilde{\mathrm{C}}^{\mathbf{C}}(\Delta)$ and $\mathrm{C}^{\mathbf{C}}(\Delta)$ with $\tilde{\mathrm{C}}(\Delta)$ and $\mathrm{C}(\Delta)$, resp., and skip the superscript, because from now on we only consider **C**-valued (self-dual) semisimple characters. We are going to apply the results of **I**, **II** and [41] to **C**-valued (self-dual) semisimple characters without further remark.

2.3. **Buildings and Moy–Prasad filtrations.** In this section we recall the description of Bruhat–Tits building of $\tilde{\mathrm{G}}$ and G and its Moy–Prasad filtrations in terms of lattice functions.

To G and $\tilde{\mathrm{G}}$ are attached Bruhat–Tits buildings $\mathfrak{B}(\mathrm{G})$, $\mathfrak{B}(\tilde{\mathrm{G}})$ and $\mathfrak{B}_{red}(\tilde{\mathrm{G}})$, see [8] and [9]. Important for the study of smooth representation, for example for the concept of depth, are the following filtrations, constructed by Moy–Prasad ([26],[27]): Let x be a point of $\tilde{\mathrm{G}}$ and y be a point of $\mathrm{B}(\mathrm{G})$. They carry

- a Lie algebra filtration
  - $(\mathfrak{g}_{y,t})_{t\in\mathbb{R}}$, $\mathfrak{g}_{y,t} \subseteq \mathrm{Lie}(\mathrm{G})$
  - $(\tilde{\mathfrak{g}}_{x,t})_{t\in\mathbb{R}}$, $\tilde{\mathfrak{g}}_{x,t} \subseteq \mathrm{Lie}(\tilde{\mathrm{G}})$

  and
- a filtration of subgroups $(\mathrm{G}_{y,t})_{t\geq 0}$, $(\tilde{\mathrm{G}}_{x,t})_{t\geq 0}$ of G and $\tilde{\mathrm{G}}$, respectively.

Those can be entirely described using lattice functions. We refer to [3], [5] and [35, §3.1] for lattice functions and lattice sequences.

Definition 2.2. A family $\Gamma = (\Gamma(t))_{t\in\mathbb{R}}$ of full $\mathfrak{o}_{\mathrm{D}}$-lattices of V is called an $\mathfrak{o}_{\mathrm{D}}$-lattice function if for all real numbers $t < s$ we have

- $\Gamma(t)$ is a full $\mathfrak{o}_{\mathrm{D}}$-lattice in V,
- $\Gamma(t) = \bigcap_{u<t} \Gamma(u)$
- $\Gamma(t)\varpi_{\mathrm{D}} = \Gamma(t + \frac{1}{d})$,

where d is the index of D (In our case of a non-split quaternion algebra $d = 2$.).

We further define $\Gamma(t+) := \bigcup_{u>t} \Gamma(u)$, and we define the set of discontinuity points:

$$\mathrm{disc}(\Gamma) := \{s \in \mathbb{R} |\ \Gamma(s) \neq \Gamma(s+)\}.$$

We can translate Γ by a real number s: $(\Gamma - s)(t) := \Gamma(t + s)$, and the set of all real translates of Γ is called the translation class of Γ. We denote this class by $[\Gamma]$. The set of all $\mathfrak{o}_{\mathrm{D}}$-lattice functions on V, resp. translation classes of those, is denoted by $\mathrm{Latt}^1_{\mathfrak{o}_{\mathrm{D}}}\mathrm{V}$, resp. $\mathrm{Latt}_{\mathfrak{o}_{\mathrm{D}}}\mathrm{V}$, see [3].

An $\mathfrak{o}_{\mathrm{D}}$-lattice function Γ with $\mathrm{disc}(\Gamma) \subseteq \mathbb{Q}$ corresponds to a lattice sequence Λ_Γ in the following way: There exists a minimal positive integer e, such that $\mathrm{disc}(\Gamma)$ is contained in $q + \frac{1}{e}\mathbb{Z}$ for some $q \in \mathbb{Q}$. Then define:

$$\Lambda_\Gamma(z) := \Gamma\left(\frac{z}{e}\right),\ z \in \mathbb{Z}.$$

Conversely we can attach a lattice function to an $\mathfrak{o}_D$-lattice sequence Λ. Recall that $\lceil t\rceil$ denotes the smallest integer not smaller than t. We define:

$$\Gamma_\Lambda(t) := \Lambda\left(\lceil te(\Lambda|F)\rceil\right),\ t\in\mathbb{R},$$

where $e(\Lambda|F)$ is the F-period of Λ.

We fix an $\mathfrak{o}_D$-lattice function Γ and an $\mathfrak{o}_D$-lattice sequence Λ. A lattice sequence Λ' is called an *affine translation* of Λ if there are a positive integer a and an integer b such that

$$\Lambda'(z) := \Lambda\left(\left\lfloor \frac{z-b}{a}\right\rfloor\right),$$

and we denote Λ' by $(a\Lambda+b)$. If $a=1$ then we call Λ' just a *translation* of Λ and we write $[\Lambda]$ for the translation class. Two lattice sequences Λ and Λ' are said to be in the same *affine class* if both have coinciding affine translations.

Then Γ and Γ_{Λ_Γ} are translates of each other and Λ and Λ_{Γ_Λ} are in the same affine class.

The invariant of the translation class of Γ is the *square lattice function*:

$$t\in\mathbb{R}\mapsto \tilde{\mathfrak{a}}_t(\Gamma) := \{a\in A|\ a\Gamma(s)\subseteq\Gamma(s+t),\ \text{for all } s\in\mathbb{R}\},\ \tilde{\mathfrak{a}}(\Gamma) := \tilde{\mathfrak{a}}_0(\Gamma),$$

and analogously we have $(\tilde{\mathfrak{a}}_z(\Lambda))_{z\in\mathbb{Z}}$ and $\tilde{\mathfrak{a}}(\Lambda)$ for $[\Lambda]$. We write $\mathrm{Latt}^2_{\mathfrak{o}_F}A$ for the set

$$\{\tilde{\mathfrak{a}}(\Gamma)|\ \Gamma\in\mathrm{Latt}^1_{\mathfrak{o}_D}V\},$$

and there are canonical maps:

$$\mathrm{Latt}^1_{\mathfrak{o}_D}V \to \mathrm{Latt}_{\mathfrak{o}_D}V \xrightarrow{\sim} \mathrm{Latt}^2_{\mathfrak{o}_F}A. \tag{2.3}$$

Note that $\mathrm{Latt}^1_{\mathfrak{o}_D}V$ carries an affine structure, see [3, § I.3]. The description of $\mathfrak{B}(\tilde{G})$ and $\mathfrak{B}_{red}(\tilde{G})$ in terms of lattice functions is stated in the following theorem.

Theorem 2.4 ([3] I.1.4,I.2.4,[8] 2.11,2.13)**.** (i) There exists an affine $\tilde{G}$-equivariant map

$$\iota_{\tilde{G}}:\ \mathfrak{B}(\tilde{G})\to\mathrm{Latt}^1_{\mathfrak{o}_D}V.$$

Further $\iota_{\tilde{G}}$ is a bijection, and two $\tilde{G}$-equivariant affine maps ι_1,ι_2 differ by a translation, i.e. there is an element $s\in\mathbb{R}$ such that $\iota_2\circ\iota_1^{-1}$ has the form

$$\Gamma\mapsto\Gamma-s.$$

(ii) There is a unique $\tilde{G}$-equivariant affine map

$$\iota_{\tilde{G},red}:\ \mathfrak{B}_{red}(\tilde{G})\to\mathrm{Latt}_{\mathfrak{o}_D}V.$$

We obtain a commutative diagram

$$\begin{array}{ccc} \mathfrak{B}(\tilde{G}) & \xrightarrow{\iota_{\tilde{G}}} & \mathrm{Latt}^1_{\mathfrak{o}_D}V \\ \downarrow & & \downarrow \\ \mathfrak{B}_{red}(\tilde{G}) & \xrightarrow{\iota_{\tilde{G},red}} & \mathrm{Latt}_{\mathfrak{o}_D}V \end{array}$$

We now describe $\mathfrak{B}(\mathrm{G})$. For more details refer to [5] and [22]. Recall the dual of a lattice function Γ:

$$\Gamma^{\#}(t) := \Gamma((-t)+)^{\#}.$$

The lattice function $\Gamma^{\#}$ depends on h, because $\#$ does, but for different ϵ-hermitian forms h_1, h_2 on V, with respect to $(\mathrm{D}, (\bar{\ }))$, with common isometry group G the respective duals $\Gamma^{\#_{h_1}}$ and $\Gamma^{\#_{h_2}}$ just differ by a translation. The lattice function Γ is called *self-dual* with respect to h if $\Gamma^{\#} = \Gamma$ or equivalently if $\tilde{\mathfrak{a}}_t(\Gamma)$ is σ_h-invariant for every $t \in \mathbb{R}$, called *self-dual* square lattice function. The map in (2.3) restricts to a canonical bijection between the set of self-dual $\mathfrak{o}_{\mathrm{D}}$-lattice functions, which we denote by $\mathrm{Latt}^1_h\mathrm{V}$, and the set of self-dual square lattice functions, denoted by $\mathrm{Latt}^2_-\mathrm{A}$. The latter inherits an affine structure from $\mathrm{Latt}_{o_{\mathrm{D}}}\mathrm{V}$.

The building of G is now described as follows in tems of lattice functions.

Theorem 2.5 ([5] 4.2)**.** There is a unique affine G-equivariant map

$$\iota_G : \mathfrak{B}(\mathrm{G}) \to \mathrm{Latt}^1_h\mathrm{V}.$$

Moreover this map is bijective.

Note that the map ι_{G} depends on h. Given (V, h_1) and (V, h_2) two ϵ-hermitian spaces w.r.t. $(\mathrm{D}, (\bar{\ }))$ with isometry group G then $\sigma_{h_1} = \sigma_{h_2}$ and we obtain the diagram

$$\begin{array}{ccc} \mathfrak{B}(\mathrm{G}) & \xrightarrow{\iota_{\mathrm{G},h_1}} & \mathrm{Latt}^1_{h_1}\mathrm{V} \\ {\scriptstyle \iota_{\mathrm{G},h_2}}\downarrow & & \downarrow\wr \\ \mathrm{Latt}^1_{h_2}\mathrm{V} & \xrightarrow{\sim} & \mathrm{Latt}^2_-\mathrm{A} \end{array}$$

and the diagram commutes by the uniqueness assertion of Theorem 2.5. Therefore for every $x \in \mathfrak{B}(\mathrm{G})$ the lattice functions $\iota_{\mathrm{G},h_1}(x)$ and $\iota_{\mathrm{G},h_2}(x)$ are in the same translation class.

Given h we embed the building $\mathfrak{B}(\mathrm{G})$ into $\mathfrak{B}(\tilde{\mathrm{G}})$ and $\mathfrak{B}_{red}(\tilde{\mathrm{G}})$ via the following diagram.

$$\begin{array}{ccccc} \mathfrak{B}(\mathrm{G}) & \longrightarrow & \mathfrak{B}(\tilde{\mathrm{G}}) & & \\ \downarrow\wr & \circlearrowleft & \downarrow\wr & & \\ \mathrm{Latt}^1_h\mathrm{V} & \longrightarrow & \mathrm{Latt}^1_{\mathfrak{o}_{\mathrm{D}}}\mathrm{V} & \longrightarrow & \mathrm{Latt}_{\mathfrak{o}_{\mathrm{D}}}\mathrm{V} \\ \downarrow\wr & \circlearrowleft & \downarrow\wr & \circlearrowleft & \downarrow\wr \\ \mathrm{Latt}^2_-\mathrm{A} & \longrightarrow & \mathrm{Latt}^2_{\mathfrak{o}_{\mathrm{F}}}\mathrm{A} & \longrightarrow & \mathfrak{B}_{red}(\tilde{\mathrm{G}}) \end{array}$$

The embedding of $\mathfrak{B}(\mathrm{G})$ into $\mathfrak{B}_{red}(\tilde{\mathrm{G}})$ does not depend on h contrary to the embedding into $\mathfrak{B}(\tilde{\mathrm{G}})$.

We now turm to the description of the Moy–Prasad filtrations. At first an $\mathfrak{o}_{\mathrm{D}}$-lattice functions Γ and an $\mathfrak{o}_{\mathrm{D}}$-lattice sequence Λ define filtrations of compact open subgroups:

$$\tilde{\mathrm{P}}_t(\Gamma) = (1 + \tilde{\mathfrak{a}}_t(\Gamma))^{\times},\ t \geq 0,$$

$$\tilde{\mathrm{P}}_n(\Lambda) = (1 + \tilde{\mathfrak{a}}_n(\Lambda))^{\times},\ n \in \mathbb{N}_0,$$

and if Γ and Λ are self-dual:

$$\mathrm{P}_t(\Gamma) := \mathrm{G}\cap\tilde{\mathrm{P}}_t(\Gamma),\ \mathrm{P}_n(\Lambda) := \mathrm{G}\cap\tilde{\mathrm{P}}_n(\Lambda),$$

and we need the filtration $\mathfrak{a}_t(\Gamma) := \tilde{\mathfrak{a}}_t(\Gamma)\cap \mathrm{A}_-$ and analogously $(\mathfrak{a}_n(\Lambda))_{n\in\mathbb{N}}$. The relation of those filtrations to the Moy–Prasad filtrations is:

Theorem 2.6 ([22])**.** If we identify the Lie algebra of $\tilde{\mathrm{G}}$ with A and the Lie algebra of G with A_-, then we have for all non-negative reel numbers t and points $x\in\mathfrak{B}(\tilde{\mathrm{G}})$ and $y\in\mathfrak{B}(\mathrm{G})$:

(i) $\tilde{\mathrm{G}}_{x,t} = \tilde{\mathrm{P}}_t(\iota_{\tilde{\mathrm{G}}}(x))$, $\tilde{\mathfrak{g}}_{x,t} = \tilde{\mathfrak{a}}_t(\iota_{\tilde{\mathrm{G}}}(x))$ and
(ii) $\mathrm{G}_{y,t} = \mathrm{P}_t(\iota_{\mathrm{G}}(y))$, $\mathfrak{g}_{y,t} = \mathfrak{a}_t(\iota_{\mathrm{G}}(y))$.

As usual we skipt the subscript zero and write $\mathrm{P}()$, $\tilde{\mathrm{P}}()$ for $\mathrm{P}_0()$, $\tilde{\mathrm{P}}_0()$.

Finally we need the description of the parahoric subgroups in terms of lattice functions/sequences. For $\tilde{\mathrm{G}}$ those are the (full) stabilizers $\tilde{\mathrm{P}}(\Gamma)$, $\tilde{\mathrm{P}}(\Lambda)$. For the classical group G the stabilizers are in general too large. The parahoric subgroup $\mathrm{P}^0(\Lambda)$ of G is constructed as follows: The quotient $\mathrm{P}(\Lambda)/\mathrm{P}_1(\Lambda)$ is the set of $\mathrm{k_F}$-rational points of a reductive group $\mathbb{P}(\Lambda)$ defined over $\mathrm{k_F}$. Let $\mathbb{P}^0(\Lambda)$ be the unital component of $\mathbb{P}(\Lambda)$. The pre-image of $\mathbb{P}^0(\Lambda)(\mathrm{k_F})$ in $\mathrm{P}(\Lambda)$ is the parahoric subgroup $\mathrm{P}^0(\Lambda)$ of G defined by Λ. Similar we have $\mathrm{P}^0(\Gamma)$ using $\mathrm{P}(\Gamma)/\mathrm{P}_{0+}(\Gamma)$.

We need a finer set of lattice functions/sequences for the study of strata.

Definition 2.7 ([3],[34] 7.1,[35] 3.6)**.** Let β be an element of $\mathrm{End_D\,V}$ and suppose $\mathrm{E} = \mathrm{F}[\beta]$ is a product of fields $\mathrm{E} = \prod_{i\in\mathrm{I}}\mathrm{E}_i$ with associated splitting $\mathrm{V} = \oplus_{i\in\mathrm{I}}\mathrm{V}^i$. An $\mathfrak{o}_\mathrm{D}$-Lattice function Γ of V is called $\mathfrak{o}_\mathrm{E}-\mathfrak{o}_\mathrm{D}$-*lattice function* if Γ is split by $(\mathrm{V}^i)_{i\in\mathrm{I}}$ and $\Gamma^i = \Gamma\cap\mathrm{V}^i$ is an $\mathfrak{o}_{\mathrm{E}_i}$-lattice function of V^i, $i\in\mathrm{I}$. Similarly we define $\mathfrak{o}_\mathrm{E}-\mathfrak{o}_\mathrm{D}$-lattice sequences.

We use $\mathrm{Latt}^1_{\mathfrak{o}_\mathrm{E},\mathfrak{o}_\mathrm{D}}\mathrm{V}$ to denote the set of $\mathfrak{o}_\mathrm{E}-\mathfrak{o}_D$-lattice functions and we set

$$\mathrm{Latt}^1_{h,\mathfrak{o}_\mathrm{E},\mathfrak{o}_\mathrm{D}}\mathrm{V} := \mathrm{Latt}^1_{\mathfrak{o}_\mathrm{E},\mathfrak{o}_\mathrm{D}}\mathrm{V}\cap\mathrm{Latt}^1_h\mathrm{V}.$$

We have two simplicial structures on $\mathfrak{B}(\mathrm{G})$. The first one, the *strong* structure, is given by the branching of $\mathfrak{B}(\mathrm{G})$, i.e. two points $x,y\in\mathfrak{B}(\mathrm{G})$ are said to lie in the same facet if, for all apartments $\mathcal{A}$ of $\mathfrak{B}(\mathrm{G})$, we have

$$x\in\mathcal{A}\text{ iff. } y\in\mathcal{A},$$

see [43]. The second simplicial structure, called *weak*, is given by intersection of facets of $\mathfrak{B}_{red}(\tilde{\mathrm{G}})$ to $\mathfrak{B}(\mathrm{G})$ via the canonical embedding of $\mathfrak{B}(\mathrm{G})$ into $\mathfrak{B}_{red}(\tilde{\mathrm{G}})$, see [1]. The strong facets are unions of weak facets and one obtains the strong structure if one removes the thin panels from the weak structure, see oriflame construction in [1].

2.4. **Centralizer.** Let Δ be a semisimple stratum and $\tilde{\mathrm{G}}_\beta$ be the centralizer of β in $\tilde{\mathrm{G}}$. Assume further that Δ is self-dual semisimple and write G_β for the centralizer of β in G. The stratum provides a pair (β,Λ) consisting of an element β of the Lie algebra of G (which generates a product E of field extensions of F) and an $\mathfrak{o}_\mathrm{E}-\mathfrak{o}_\mathrm{D}$-lattice sequence Λ.

We need to attach to them a point of $\mathfrak{B}(G_\beta)$ ($\cong \prod_{i\in I_{0,+}} \mathfrak{B}(G^i_\beta)$, G^i the image of the projection of G to $\mathrm{Aut}_D(V^i)$), and interpret this point as a tuple of lattice sequences Λ^i_β, $i \in I$. The important requirement the tuple has to satisfy is the compatibility with the Lie algebra filtrations (abv. **CLF**, cf. [33, §6]):

$$\mathfrak{a}_z(\Lambda) \cap \mathrm{End}_{E\otimes_F D}(V) = \oplus_{i\in I_0} \mathfrak{a}_z(\Lambda^i_\beta) \oplus_{i\in I_+} \tilde{\mathfrak{a}}_z(\Lambda^i_\beta),\ z \in \mathbb{Z}. \tag{2.8}$$

The CLF-property is meant with respect to the canonical embedding of Lie algebras $\prod_{i\in I_{0,+}} \mathrm{End}_{E_i\otimes D} V^i \longrightarrow \mathrm{End}_D V$ (no I_-, and canonical with respect to h.) We also need the CLF-property for general linear groups:

$$\tilde{\mathfrak{a}}_z(\Lambda) \cap \mathrm{End}_{E\otimes_F D}(V) = \oplus_{i\in I} \tilde{\mathfrak{a}}_z(\Lambda^i_\beta),\ z \in \mathbb{Z}. \tag{2.9}$$

The construction of $(\Lambda^i_\beta)_{i\in I}$ is done in several steps:

Remark 2.10. We interpret the buildings $\mathfrak{B}(G_\beta)$ and $\mathfrak{B}(G)$ in terms of lattice functions using §2.3. Let Γ_Λ be the lattice function attached to Λ.

(i) By [33, Theorem 7.2] there exists a G_β-equivariant, affine, injective CLF-map

$$j_\beta : \mathfrak{B}(G_\beta) \hookrightarrow \mathfrak{B}(G)$$

whose image in terms of lattice functions is $\mathrm{Latt}^1_{h,\mathfrak{o}_E,\mathfrak{o}_D} V$, in particular it contains Γ_Λ.

(ii) We define for $i \in I$ the skewfields:

$$D^i_\beta := \begin{cases} E_i \otimes D, & \text{if } E_i \text{ has odd degree over F.} \\ E_i, & \text{else} \end{cases}.$$

and there is a right-D^i_β-vector space V^i_β such that $\mathrm{End}_{D^i_\beta}(V^i_\beta)$ is E_i-algebra isomorphic to $\mathrm{End}_{E_i\otimes D}(V^i)$, and further we can find for every $i \in I_0$ an ϵ-herrmitian-D^i_β-form h^i_β on V^i_β such that its adjoint anti-involution $\sigma_{h^i_\beta}$ coincides with the pullback of the restriction of σ_h. The construction of j_β in *loc.cit.* (which mainly uses [3, II.1.1.]) provides a tuple $(\Gamma^i_\beta)_{i\in I_{0.+}}$ of $\mathfrak{o}_{D^i_\beta}$-lattice functions, such that $j_\beta((\Gamma^i_\beta)_{i\in I_{0.+}}) = \Gamma_\Lambda$. We fix a map [3, II.3.1] and attach an $\mathfrak{o}_{D^i_\beta}$-lattice function Γ^i_β to $\Gamma \cap V^i$ for $i \in I_-$.

(iii) Let e be the F-period of Λ. We define the $\mathfrak{o}_{D^i_\beta}$-lattice sequence Λ^i_β via $\Lambda^i_\beta(z) := \Gamma^i_\beta(\frac{z}{e})$.

(iv) Both CLF-properties (2.8) and (2.9) are satisfied, see [34, Theorem 7.2] and [3, Theorem II.1.1].

We write Λ_β for $(\Lambda^i_\beta)_{i\in I}$ and we are going to write $\mathfrak{b}(\Lambda), \tilde{\mathfrak{b}}(\Lambda), \mathfrak{b}_z(\Lambda), \tilde{\mathfrak{b}}_z(\Lambda)$ for the intersections of $\mathfrak{a}(\Lambda), \tilde{\mathfrak{a}}(\Lambda), \mathfrak{a}_z(\Lambda), \tilde{\mathfrak{a}}_z(\Lambda)$ with B. Similarly we define $\tilde{P}(\Lambda_\beta)$, $P(\Lambda_\beta)$, etc. as the intersection with $\tilde{G}_\beta$ of the respective objects, except $P^0(\Lambda_\beta)$, which we define as the parahoric subgroup of G_β attached to Λ_β, in particular $P^0(\Lambda_\beta) \cong \prod_{i\in I_{0,+}} P^0(\Lambda^i_\beta)$.

2.5. **Base extension.** Consider $\mathbb{G}$ from §2.1. Let L|F be a quadratic unramified extension of F. In fact we only will use the carefully chosen extension in **II**.2.1. There is a very explicit description of the base extension from F to L in §2 of **II**. Later we are going to reduce statements to the group $\mathbb{G}(\mathrm{L})$, also denoted by $\mathrm{G}\otimes\mathrm{L}$, in particular this includes the condition det = 1. We will use a canonical injective map i_{L} of $\mathfrak{B}(\mathrm{G})$ into $\mathfrak{B}(\mathrm{G}\otimes\mathrm{L})$ given by $\Gamma_{i_{\mathrm{L}}(x)}=\Gamma_x$. and the same for the ambient general linear groups: $i_{\mathrm{L}}:\mathfrak{B}(\tilde{\mathrm{G}})\to\mathfrak{B}(\tilde{\mathrm{G}}\otimes\mathrm{L})$. If we work over L we give the objects in question the subscript L, for example we write $\tilde{\mathfrak{g}}_{\mathrm{L},x}$ and $\mathfrak{g}_{\mathrm{L},x}$ for the Moy–Prasad filtration of a point x in $\mathfrak{B}(\mathrm{G}\otimes\mathrm{L})$. For the definition of semisimple characters of $\tilde{\mathrm{G}}\otimes\mathrm{L}$ we choose the $\mathrm{Gal}(\mathrm{L}|\mathrm{F})=\langle\tau\rangle$-fixed extension ψ_{L} of ψ_{F} given by $\psi_{\mathrm{L}}(x):=\psi_{\mathrm{F}}(\frac{1}{2}\mathrm{tr}_{\mathrm{L}|\mathrm{F}}(x))$. There is an action of Gal(L|F) on $\mathfrak{B}(\tilde{\mathrm{G}}\otimes\mathrm{L})$. It induces a Gal(L|F)-action on $\mathfrak{B}_{red}(\tilde{\mathrm{G}}\otimes\mathrm{L})$ and $\mathfrak{B}(\mathrm{G}\otimes\mathrm{L})$. The action on $\mathfrak{B}(\tilde{\mathrm{G}}\otimes\mathrm{L})$ is defined by

$$(\tau.\Gamma_{\mathrm{L}})(t):=(\Gamma_{\mathrm{L}})(t-\frac{1}{2})\varpi_{\mathrm{D}}.$$

Further the Gal(L|F)-action on $\mathrm{A}\otimes_{\mathrm{F}}\mathrm{L}=\mathrm{End}_{\mathrm{L}}\mathrm{V}$ is defined by the Gal(L|F)-action on the second factor. The latter action coincides with the conjugation with $\varpi_{\mathrm{D}}\in\mathrm{End}_{\mathrm{F}}\mathrm{V}$.

2.6. **Intertwining.** We recall the notions of intertwining. Suppose we are given a smooth representation γ on some compact open subgroup K of some totally disconnected locally compact group H. For an element $g\in\mathrm{H}$ we write $\mathrm{I}_g(\gamma)$ for $\mathrm{Hom}_{\mathrm{K}\cap\mathrm{K}^g}(\gamma,\gamma^g)$, and we denote by $\mathrm{I}_{\mathrm{H}}(\gamma)$ the set of all $g\in\mathrm{H}$ such that $\mathrm{I}_g(\gamma)$ is non-zero, and we call $\mathrm{I}_{\mathrm{H}}(\gamma)$ the set of *intertwining elements* of γ in H.

2.7. **Restriction.** We recall that, given locally compact totally disconnected groups G_1 and G_2 such that G_2 is a topological subgroup of G_1, we denote by $\mathrm{Res}^{\mathrm{G}_1}_{\mathrm{G}_2}$ the functor from $\mathfrak{R}(\mathrm{G}_1)$ to $\mathfrak{R}(\mathrm{G}_2)$ given by restriction from G_1 to G_2.

2.8. **Endomorphisms and involutions.** We have the following algebras of endomorphisms:

$$\mathrm{A}=\mathrm{End}_{\mathrm{D}}\mathrm{V},\ \mathrm{A}_{\mathrm{F}}=\mathrm{End}_{\mathrm{F}}\mathrm{V},\ \mathrm{A}_{\mathrm{L}}=\mathrm{End}_{\mathrm{L}}\mathrm{V},$$

and they can be interpreted as fixed point sets using involutions. As already fixed as a notation we have that l is a τ-skew-symmetric element of $\mathfrak{o}_{\mathrm{L}}^{\times}$ and ϖ_{D} is a uniformizer of D, and both have the poperty that their square is an element of F. Both elements define F-endomorphisms of V via right-multiplication and we still denote these endomorphisms by l and ϖ_{D}. Let τ_l and $\tau_{\varpi_{\mathrm{D}}}$ be the involutions on A_{F} given by conjugation with l and ϖ_{D}, respectively. Then both generate a Kleinian four-group $\langle\tau_l,\tau_{\varpi_{\mathrm{D}}}\rangle$ whose fixed point set is A. The averaging function

$$\varnothing_{\langle\tau_l,\tau_{\varpi_{\mathrm{D}}}\rangle}:\ \mathrm{A}_{\mathrm{F}}\twoheadrightarrow\mathrm{A}$$

defined via

$$\varnothing_{\langle\tau_l,\tau_{\varpi_{\mathrm{D}}}\rangle}(x):=\frac{1}{4}(x+\tau_l(x)+\tau_{\varpi_{\mathrm{D}}}(x)+\tau_l(\tau_{\varpi_{\mathrm{D}}}(x)))$$

is a very useful projector for reducing proofs from D to F.

2.9. **Tame corestriction.** The aim of this paragraph is to show the construction of self-dual tame corestrictions over D. This should already have been done in **II**, and one could leave it as an exercise for the reader, but since it is important in the proof of Theorem 3.1 and to indicate the little difference in the construction between the skew and the self-dual case, we have decided to devote to it a paragraph. It is an almost trivial generalization of [41, Proof of 3.31] and [31, 4.13], all based on [11, (1.3.4)].

For the GL-case tame corestrictions in the semisimple case are already defined in **I**4.13. Given a semisimple stratum $\Delta = [\Lambda, n, r, \beta]$ then a map $s : \mathrm{A} \to \mathrm{B}$ is called tame corestriction for β if under the canonical isomorphism $\mathrm{A}_{\mathrm{F}} \simeq \mathrm{A} \otimes_{\mathrm{F}} \mathrm{End}_{\mathrm{A}}(\mathrm{V})$ the map $s_{\mathrm{F}} = s \otimes \mathrm{id}_{\mathrm{End}_{\mathrm{A}}(\mathrm{V})}$ is a tame corestriction in the sense of [36, 6.17].

Definition 2.11. We call a tame corestriction s *self-dual* (with respect to h) if s is σ_h-invariant.

A corestriction for β always exists by [6, 4.2.1], and we fix one, say s. We define $h_{\mathrm{F}} = \mathrm{trd}_{\mathrm{D}|\mathrm{F}} \circ h$ whose anti-involution $\sigma_{h_{\mathrm{F}}}$ extends σ_h from A to A_{F}. Suppose at first Δ is simple. Two tame corestrictions $s_{1,\mathrm{F}}$ and $s_{2,\mathrm{F}}$ differ by a multiplication with an element of $\mathfrak{o}_{\mathrm{E}}^{\times}$, see [11, (1.3.4)]. Then, as $\mathfrak{o}_{\mathrm{E}}^{\times}$ is contained in A, every multiplication of s by an element u of $\mathfrak{o}_{\mathrm{E}}^{\times}$ provides a tame corestriction. If s_{F} is self-dual, so is s too, because A and $\mathrm{End}_{\mathrm{A}}(\mathrm{V})$ are $\sigma_{h_{\mathrm{F}}}$-invariant. We conclude the existence of a self-dual s from *loc.cit.* in taking a σ_h-invariant additive character ψ_{E}. For the semisimple case we choose a corestriction $s_i : \mathrm{A}^{ii} \to \mathrm{B}^i$ for every $i \in \mathrm{I}_0 \cup \mathrm{I}_+$ (self-dual for $i \in \mathrm{I}_0$) and define

$$s_{-i}(a) \coloneqq s_i(\sigma_h(a)),\ a \in \mathrm{A}^{ii},\ i \in \mathrm{I}_- .$$

Finally, the definition:

$$s(a) \coloneqq \sum_{i \in \mathrm{I}} s_i(a^{ii}),\ a \in \mathrm{A},$$

provides a self-dual corestriction.

2.10. **Brauer characters and Glauberman correspondence.** Most of the statements in the construction of cuspidal representations for locally compact totally disconnected classical groups are first written for complex representations and referred via Brauer characters to the modular case. The objects modular and complex in the construction as for example various Heisenberg representations for semisimple characters correspond to each other via the Brauer map. The objects are defined starting from semisimple characters followed by uniqueness statements using restriction and induction. The Brauer map preserves those operations which implies the above claim.

In contrast to the rest, in this section if we say *character* we mean the trace of a representation. Let K be a finite group. Here in this section we forget about the category structure of $\mathfrak{R}_{\mathbf{C}}(\mathrm{K})$ and consider it just as the set of isomorphism classes of $\mathbf{C}$-representations of K. We further assume for this paragraph that $p_{\mathbf{C}}$ is positive and does not divide the order of K.

The following construction is provided in [28, §2]. In choosing a maximal ideal containing $p_{\mathbf{C}}$ in the ring of algebraic integers they fix a group isomorphism between the groups of roots of unity of order prime to $p_{\mathbf{C}}$

$$()^* : \mu_{p_{\mathbf{C}}\prime}(\mathbb{C}) \xrightarrow{\sim} \mu_{p_{\mathbf{C}}\prime}(\mathbf{C}), \tag{2.12}$$

and extend this map to the additive closure (in fact it is extended to the localization of the ring of algebraic integers with respect to the mentioned maximal ideal.) Using (2.12) one attaches to a representation $\eta \in \mathfrak{R}_{\mathbf{C}}(\mathrm{K})$ a complex character χ_η, the Brauer character of η, and therefore a representation $\mathrm{Br}_{\mathrm{K}}(\eta) \in \mathfrak{R}_{\mathbb{C}}(\mathrm{K})$, and this definition is compatible with direct sums and restriction ($\mathrm{K} \geq \mathrm{K}'$):

$$\begin{array}{ccc} \mathfrak{R}_{\mathbf{C}}(\mathrm{K}) & \stackrel{\mathrm{Br}_{\mathrm{K}}}{\longrightarrow} & \mathfrak{R}_{\mathbb{C}}(\mathrm{K}) \\ \mathrm{Res}^{\mathrm{K}}_{\mathrm{K}'} \downarrow & \circlearrowleft & \downarrow \mathrm{Res}^{\mathrm{K}}_{\mathrm{K}'} \\ \mathfrak{R}_{\mathbf{C}}(\mathrm{K}') & \stackrel{\mathrm{Br}_{\mathrm{K}'}}{\longrightarrow} & \mathfrak{R}_{\mathbb{C}}(\mathrm{K}'). \end{array} \tag{2.13}$$

We call Br_{K} the *Brauer map* for K. Note that the map $(\chi_\eta)^*$ given by $(\chi_\eta)^*(g) := (\chi_\eta(g))^*$, $g \in \mathrm{K}$, is the character of η with values in $\mathbf{C}$ given by the trace.

Theorem 2.14 ([28, p18, Theorems 2.6 and 2.12])**.** The Brauer map for K is a bijection, preserves dimensions and maps the set of classes of irreducible $\mathbf{C}$-representations onto the set of classes of irreducible $\mathbb{C}$-representations of K.

In this article we only will apply Br_{K} for nilpotent groups. In that case it follows immediately that Br commutes with induction.

Proposition 2.15. Suppose K is nilpotent. Then the following diagram is commutative.

$$\begin{array}{ccc} \mathfrak{R}_{\mathbf{C}}(\mathrm{K}) & \stackrel{\mathrm{Br}_{\mathrm{K}}}{\longrightarrow} & \mathfrak{R}_{\mathbb{C}}(\mathrm{K}) \\ \mathrm{ind}^{\mathrm{K}}_{\mathrm{K}'} \uparrow & \circlearrowleft & \uparrow \mathrm{ind}^{\mathrm{K}}_{\mathrm{K}'} \\ \mathfrak{R}_{\mathbf{C}}(\mathrm{K}') & \stackrel{\mathrm{Br}_{\mathrm{K}'}}{\longrightarrow} & \mathfrak{R}_{\mathbb{C}}(\mathrm{K}'). \end{array} \tag{2.16}$$

Proof. By transitivity of induction it is enough to consider a normal subgroup K' of K, and we will do so from now on in the proof. For $\eta' \in \mathfrak{R}_{\mathbf{C}}(\mathrm{K}')$ let χ' be the character of $\mathrm{Br}_{\mathrm{K}'}(\eta')$ and χ be the character of $\mathrm{ind}^{\mathrm{K}}_{\mathrm{K}'} \mathrm{Br}_{\mathrm{K}'}(\eta)$. Now they satisfy:

$$\chi(g) = \begin{cases} \sum_{\bar{k} \in \mathrm{K}/\mathrm{K}'} ({}^k\chi')(g) & , \ g \in \mathrm{K}' \\ 0 & , \ g \notin \mathrm{K}' \end{cases}$$

Thus χ^* is the $\mathbf{C}$-character of $\mathrm{ind}^{\mathrm{K}}_{\mathrm{K}'} \eta'$. Thus the diagram (2.16) is commutative. □

One can now transfer the Glauberman correspondence [18] to the modular case via the Brauer map.

Let Γ be a finite solvable operator group acting on K such that K and Γ have relatively prime orders. Let K^Γ be the fixed point set of K. We consider now sets of isomorphism classes of irreducible representation denoted by $\mathrm{Irr}_?(!)$. The group Γ acts on $\mathrm{Irr}_?(\mathrm{K})$ and the fixed point set will be denoted by $\mathrm{Irr}_?(\mathrm{K})^\Gamma$. Then Glauberman constructs a bijection

$$\mathbf{gl}^\Gamma_{\mathbb{C}} : \ \mathrm{Irr}_{\mathbb{C}}(\mathrm{K})^\Gamma \xrightarrow{\sim} \mathrm{Irr}_{\mathbb{C}}(\mathrm{K}^\Gamma),$$

and we define the *Glauberman transfer* $\mathbf{gl}^\Gamma_{\mathbf{C}}$ for $\mathbf{C}$-representations via $\mathbf{gl}^\Gamma_{\mathbf{C}} := \mathrm{Br}^{-1}_{\mathrm{K}^\Gamma} \circ \mathbf{gl}^\Gamma_{\mathbb{C}} \circ \mathrm{Br}_{\mathrm{K}}|_{\mathrm{Irr}_{\mathbf{C}}(\mathrm{K})^\Gamma}$.

If Γ is the Galois group of the field extension L|F then we write $\mathbf{gl}_{\mathbf{C}}^{\mathrm{L|F}}$ for the Glauberman transfer.

Remark 2.17. In this work we only apply the Glauberman correspondence for Γ a cyclic group of order two. So here $\mathbf{gl}_?(\eta)$ is the unique element of $\mathrm{Irr}_?(\mathrm{K}^\Gamma)$ with odd multiplicity in $\mathrm{Res}^{\mathrm{K}}_{\mathrm{K}^\Gamma}(\eta)$.

We set for $\mathbf{C}$ with characteristic zero Br_{K} to be the identity map.

3. Exhaustion for semisimple Characters

In this section we prove the following theorem.

Theorem 3.1 (see [41] 5.1 and [14] §8.9 for the non-quaternionic case)**.** Let π be an irreducible representation of G. Then there is a self-dual semisimple stratum Δ with $r = 0$ and an element θ of $\mathrm{C}(\Delta)$ such that θ is contained in π.

The proof of this theorem requires several steps similar to *loc.cit.*. We fix an irreducible smooth representation π of G. The proof of the theorem is done by induction.

(i) In the base case we have the existence of a trivial semisimple character of the same depth as π contained in π.
(ii) The induction for $\theta \in \mathrm{C}(\Delta)$ contained in π is on the fraction $\frac{r}{e(\Lambda|F)}$.
(iii) For the induction step we need to be able to change lattice sequences: Roughly speaking, given a self-dual semisimple character $\theta \in \mathrm{C}(\Delta)$ with positive r and contained in π there is a self-dual stratum Δ' such that $\beta = \beta'$ and $\frac{r'}{e(\Lambda'|F)} < \frac{r}{e(\Lambda|F)}$ and an element $\theta' \in \mathrm{C}(\Delta')$ which is contained in π.
(iv) These steps are not enough, because one has to ensure that the difference between $\frac{r}{e(\Lambda|F)}$ and $\frac{r'}{e(\Lambda'|F)}$ is bounded from below by a positive constant independent of Λ and Λ'.

Recall that the depth of π is the infimum of all non-negative real numbers t with a point x in $\mathfrak{B}(\mathrm{G})$ such that the trivial representation of $\mathrm{G}_{x,t+}$ is contained in π. Let us recall that the barycentric coordinates of a point x in $\mathfrak{B}(\mathrm{G})$ are the barycentric coordinates of the point with respect to the vertexes of any chamber containing x.

Lemma 3.2 (see [26] 5.3, 7.4)**.** The depth of π is attained at a point with rational barycentric coordinates.

This follows from *loc.cit.* because the depth is attained at an optimal point, see *loc.cit.* 7.4.

Remark 3.3. One could think that a continuity argument with the function

$$x \in \mathfrak{B}(\mathrm{G}) \mapsto \mathrm{d}(\pi, x) := \inf\{s \geq 0 |\ V_\pi^{\mathrm{G}_{x,s+}} \neq 0\}$$

could lead to Lemma 3.2, but it is unclear if this function is continuous. It is upper-continuous, but maybe not lower continuous. Here the idea in *loc.cit.* of taking optimal points comes into play which form a finite set for a given chamber C.

We prove the upper-continuity of $\mathrm{d}(\pi,*)$ in the above remark. It is not needed for what follows in this article.

Proof. Take $x\in\mathfrak{B}(\mathrm{G})$. It corresponds to a self-dual lattice function Γ_x with set $\mathrm{disc}(x)$ of discontinuity points of Γ. We take a CAT(0)-metric $d(*,*)$ on $\mathfrak{B}(\mathrm{G})$ given in [43]. The point x lies in the interior of a facet F_x. Then for all positive real δ_1 there exists a positive real δ_2 such that

(i) The ball around x with radius δ_2 does not intersect any facet of lower dimension than the dimension of F_x.
(ii) For all $x'\in\mathfrak{B}(\mathrm{G})$ with $d(x,x')<\delta_2$ and all $t\in\mathrm{disc}(x)$ there exists a $t'\in\mathrm{disc}(x')$ such that $|t-t'|<\delta_1$ and $\Gamma_x(t)=\Gamma_{x'}(t')$.
(iii) For all $x'\in\mathfrak{B}(\mathrm{G})$ with $d(x,x')<\delta_2$ and all $t'\in\mathrm{disc}(x')$ there exists a $t\in\mathrm{disc}(x)$ such that $|t-t'|<\delta_1$ and $\Gamma_x(t)\supseteq\Gamma_{x'}(t')$.

Statement (iii) implies $\Gamma_x(t)\supseteq\Gamma_{x'}(t+\delta_1)$ for all $t\in\mathbb{R}$. Then, for every $x'\in\mathfrak{B}(\mathrm{G})$ with $d(x,x')<\delta_2$ and every $s\geq 0$, the group $\mathrm{G}_{x,s+}$ contains $\mathrm{G}_{x',(s+2\delta_1)+}$. In particular those x' satisfy $\mathrm{d}(\pi,x')\leq\mathrm{d}(\pi,x)+2\delta_1$ which finishes the proof. □

For the proofs of the next lemmas we need some duality for the Moy–Prasad filtrations. Given a subset S of A the dual S^* of S with respect to ψ_{F} is defined as the subset of A consisting of all elements a of A which satisfy $\psi_A(sa)=1$ for all $s\in S$. ($\psi_{\mathrm{A}}:=\psi_{\mathrm{F}}\circ\mathrm{trd}$). The main property of this duality is:

Lemma 3.4. Let x be a point of $\mathfrak{B}(\tilde{\mathrm{G}})$. Then we have $\tilde{\mathfrak{g}}_{x,t}^*=\tilde{\mathfrak{g}}_{x,-t+}$ for all $t\in\mathbb{R}$.

Proof. We choose a splitting basis $(v_k)_k$ for Γ_x, i.e.

$$\Gamma_x(t)=\oplus_k v_k\mathfrak{p}_{\mathrm{D}}^{\lceil d(t-a_k)\rceil}.$$

Depending on the choice of the basis we have an F-algebra isomorphism $\mathrm{A}\simeq\mathrm{M}_m(\mathrm{D})$ and therefore a left D-vector space action on A given by the canonical one on $\mathrm{M}_m(\mathrm{D})$. For $1\leq j,k\leq m$ let E_{ij} be the element of A with kernel $\oplus_{k\neq j}v_{\mathrm{k}}\mathrm{D}$ which sends v_j to v_i. They form a D-left basis of A which splits $\tilde{\mathfrak{g}}_x$, more precisely:

$$\tilde{\mathfrak{g}}_{x,t}=\oplus_{ij}\mathfrak{p}_{\mathrm{D}}^{\lceil d(t+a_j-a_i)\rceil}E_{ij}.$$

We now show the assertion of the lemma. The inclusion $\supseteq$ is obvious. For the other inclusion we first remark that for every reel number t the lattice $\tilde{\mathfrak{g}}_{x,t}^*$ is split by $(E_{ij})_{ij}$ too:

$$\tilde{\mathfrak{g}}_{x,t}^*=\oplus_{ij}\mathfrak{p}_{\mathrm{D}}^{e_{ij}(t)}E_{ij},\ e_{ij}(t)\in\mathbb{Z}$$

Then $e_{ji}(t)+\lceil d(t+a_j-a_i)\rceil$ is positive and therefore $e_{ji}(t)>d(-t+a_i-a_j)$ which finishes the proof since $e_{ji}(t)$ is an integer. □

For an element $\beta\in\mathrm{A}$ we define the map $\tilde{\psi}_\beta:\mathrm{A}\to\mathbf{C}$ via $\tilde{\psi}_\beta(1+a):=\psi_{\mathrm{A}}(\beta a)$. Some restrictions of $\tilde{\psi}_\beta$ are characters, i.e. multiplicative, as for example in the following case:

Definition 3.5. Let $\Delta = [\Lambda, n, n-1, \beta]$ be a stratum which is not equivalent to a null-stratum. Then we define $d_\Delta := \frac{n}{e(\Lambda|F)}$ to be the depth of Δ. Let $x \in \mathfrak{B}(\tilde{\mathrm{G}})$ be a point corresponding to Λ. The coset of Δ in terms of the building is defined as $\beta + \tilde{\mathfrak{g}}_{x,-d_\Delta+}$, and if Δ is self-dual then we call $\beta + \mathfrak{g}_{x,-d_\Delta+}$ its *self-dual* coset. To Δ is attached the character $\tilde{\psi}_\Delta : \tilde{\mathrm{G}}_{x,d_\Delta} \to \mathbf{C}$ defined via restriction of $\tilde{\psi}_\beta$. Note that $\tilde{\psi}_\Delta$ is trivial on $\tilde{\mathrm{G}}_{x,d_\Delta+}$ by Lemma 3.4. If Δ is self-dual we write ψ_Δ for the restriction of $\tilde{\psi}_\Delta$ to G_{x,d_Δ}. We say that π contains Δ or the associate coset, if it contains ψ_Δ.

Proposition 3.6 (cf. [40] 2.11 and 2.13 for G⊗L)**.** Suppose π has positive depth. Then π contains a self-dual semisimple stratum Δ with $n = r+1$ and of the same depth.

For the proposition we need a convexity lemma for semisimple strata. Let us recall: The minimal polynomial μ_Δ for a stratum $\Delta := [\Lambda, n, n-1, \beta]$ is the minimal polynomial in $\mathrm{k_F}[X]$ of the residue class $\bar{\eta}(\Delta)$ of $\eta(\Delta) := \varpi_{\mathrm{F}}^{\frac{n}{\gcd(n,e(\Lambda|F))}} \beta^{\frac{e(\Lambda|F)}{\gcd(n,e(\Lambda|F))}}$ modulo $\tilde{\mathfrak{a}}_1(\Lambda)$, see **I**.§4.2. Further we recall the *characteristic polynomial* χ_Δ of Δ which is the mod $\mathfrak{p}_{\mathrm{F}}$-reduction of the reduced characteristic polynomial of $\eta(\Delta)$. The polynomials μ_Δ and χ_Δ depend on the choice of ϖ_{F}. For the next lemma we need the *barycentre* $\frac{1}{2}\Lambda + \frac{1}{2}\Lambda'$ of two lattice sequences Λ, Λ' with the same F-period e:

$$\left(\frac{1}{2}\Lambda + \frac{1}{2}\Lambda'\right)(m) := \left(\frac{1}{2}\Gamma_\Lambda + \frac{1}{2}\Gamma_{\Lambda'}\right)\left(\frac{m}{2e}\right),\ m \in \mathbb{Z}.$$

Here we have used the affine structure on $\mathrm{Latt}^1_{\mathfrak{o}_\mathrm{D}}\mathrm{V}$, see the description in [5, Proof of 7.1].

Lemma 3.7. Suppose that $\Delta := [\Lambda, n, n-1, \beta]$ and $\Delta' := [\Lambda', n, n-1, \beta]$ are strata over D which are equivalent to semisimple strata and share the characteristic polynomial and the D-period. Then $\Delta'' := [\frac{1}{2}\Lambda + \frac{1}{2}\Lambda', 2n, 2n-1, \beta]$ is equivalent to a semisimple stratum.

Proof. It suffices to prove that $\mathrm{Res_F}(\Delta'')$ is equivalent to a semisimple stratum, by [35, Theorem 4.54], and therefore for the proof we suppose that Δ, Δ' are strata over F instead of D. At first Δ and Δ' have the same minimal polynomial because it is the radical of the characteristic polynomial by the equivalence to semisimple strata. By convexity, see [40, 5.5], we obtain that the minimal polynomial of Δ'' divides μ_Δ. But since Δ is equivalent to a semisimple stratum we obtain by **I**.4.8 that μ_Δ and therefore $\mu_{\Delta''}$ is square-free. Thus, in case that X does not divide $\mu_{\Delta''}$, we are done by **I**4.8.

Suppose now that X is a divisor of μ_Δ. The element $\bar{\eta}(\Delta)$ generates a semisimple algebra over $\mathrm{k_F}$ which is isomorphic to the algebra generated by $\bar{\eta}(\Delta')$ via $\eta(\Delta)$ mod $\tilde{\mathfrak{a}}_1(\Lambda)$ is send to $\eta(\Delta')$ mod $\tilde{\mathfrak{a}}_1(\Lambda')$, see [35, 4.46]. Let e be an idempotent which splits Λ and commutes with β and such that the minimal polynomial of the stratum $e\Delta := [e\Lambda, n, n-1, e\beta]$ is X and $X \nmid \mu_{(1-e)\Delta}$. See [36, 6.11] for the construction. The element e is a polynomial in $\eta(\Delta)$ (Note: $\eta(\Delta) = \eta(\Delta')$) with coefficients in $\mathfrak{o}_\mathrm{F}$. Thus e also splits Λ'. Further $e\Delta$ and $e\Delta'$ intertwine which implies that $e\Delta'$ cannot be fundamental by [36, 6.9], i.e. the square-free minimal polynomial of $e\Delta'$ needs to be just X. The stratum $(1-e)\Delta'$ is fundamental and has the same minimal polynomial as $(1-e)\Delta$, because the second is fundamental and both strata intertwine. Thus $(1-e)\Delta''$ is equivalent to a semisimple stratum by the first part of the proof. This reduces to the case $e = 1$, i.e. Δ and Δ' are equivalent to null-strata. But then Δ'' is equivalent to a null-stratum too, by convexity [40, 5.5]. This finishes the proof. □

For the proof of Proposition 3.6 we are going to use Theorem B.1 (cf. [40, Theorem 4.4]) whose proof uses the erratum of [40, Proposition 4.2] in §A, see Proposition A.1.

Proof of Proposition 3.6. The depth of π is rational because, by [26], π attains its depth at an optimal point of $\mathfrak{B}(\mathrm{G})$, say x, in particular at a point with rational barycentric coordinates. Then there is an element b of $\mathfrak{g}_{x,-d_\pi}$ such that $b+\mathfrak{g}_{x,-d_\pi+}$ is contained in π. We are going to show that there is a self-dual semisimple stratum Δ with $r+1=n$ whose coset (in $\tilde{\mathfrak{g}}$) contains the coset $b+\tilde{\mathfrak{g}}_{x,-d_\pi+}$. By Theorem B.1 there is a point $x'_{\mathrm{L}}\in\mathfrak{B}(\mathrm{G}\otimes\mathrm{L})$ with rational barycentric coordinates which satisfies the following property $(*)$: The coset $b+\tilde{\mathfrak{g}}_{x'_{\mathrm{L}},-d_\pi+}$ is a coset of a semisimple stratum over L, $b\in\tilde{\mathfrak{g}}_{x'_{\mathrm{L}},-d_\pi}$ and $\tilde{\mathfrak{g}}_{x'_{\mathrm{L}},-d_\pi+}$ contains $\tilde{\mathfrak{g}}_{i_{\mathrm{L}}(x),-d_\pi+}$. The Galois group $\langle\tau\rangle$ of L|F acts on $\mathfrak{B}(\tilde{\mathrm{G}}\otimes\mathrm{L})$ with fixed point set $\mathfrak{B}(\tilde{\mathrm{G}})$, and $\tau(x'_{\mathrm{L}})$ also satisfies property $(*)$. We define x''_{L} as the barycentre of the segment between x'_{L} and $\tau(x'_{\mathrm{L}})$. Then x''_{L} also satisfies $(*)$. Indeed: the containments are trivial by convexity [41, 5.5], and the corresponding coset is a coset of a semisimple stratum, by Lemma 3.7. The point x''_{L} is fixed by τ and is therefore of the form $i_{\mathrm{L}}(x'')$ for some $x''\in\mathfrak{B}(\mathrm{G})$. Let $\Delta''=[\Lambda'',n'',n''-1,b]$ be a self-dual stratum for the coset $b+\tilde{\mathfrak{g}}_{x'',-d_\pi+}$. Note that Δ'' is not equivalent to a null-stratum, i.e. $b\notin\tilde{\mathfrak{g}}_{x'',-d_\pi+}$, by the definition of d_π, as the coset $b+\mathfrak{g}_{x'',-d_\pi+}$ is contained in π. In other words: the depth of Δ'' is d_π. As $\Delta''\otimes\mathrm{L}$ is equivalent to a semisimple stratum and as $b\in\mathrm{Lie}(\mathrm{G})$ we obtain that Δ'' is equivalent to a self-dual semisimple stratum by **I**4.54 and **II**.4.7. □

The same idea of base extension using Theorem B.1 shows:

Corollary 3.8 (cf. [40] 4.4)**.** Let Δ be a self-dual stratum with $n=r+1$.

(i) If Δ is fundamental then there exists a self-dual semisimple stratum Δ' with $n'=r'+1$ such that $\frac{n}{e(\Lambda|F)}=\frac{n'}{e(\Lambda'|F)}$, and $\beta+\tilde{\mathfrak{a}}_{-r}\subseteq\beta'+\tilde{\mathfrak{a}}'_{-r'}$.

(ii) If Δ is not fundamental, then there exists a self-dual null-stratum $[\Lambda',r',r',0]$ such that
$$\beta+\tilde{\mathfrak{a}}_{-r}(\Lambda)\subseteq\tilde{\mathfrak{a}}'_{-r'}(\Lambda')$$
and $\frac{r'}{e(\Lambda'|\mathrm{F})}$ is smaller than $\frac{n}{e(\Lambda|\mathrm{F})}$

Apparently we also need the $\tilde{\mathrm{G}}$-version, whose proof is similar to the proof of the previous corollary, using the $\tilde{\mathrm{G}}$-part of Theorem B.1 instead.

Corollary 3.9 (For (ii) cf. [31] 3.11)**.** Let Δ be a stratum with $n=r+1$.

(i) If Δ is fundamental then there exists a semisimple stratum Δ' with $n'=r'+1$ such that $\frac{n}{e(\Lambda|F)}=\frac{n'}{e(\Lambda'|F)}$, and $\beta+\tilde{\mathfrak{a}}_{-r}\subseteq\beta'+\tilde{\mathfrak{a}}'_{-r'}$.

(ii) If Δ is not fundamental, then there exists a null-stratum $[\Lambda',r',r',0]$ such that
$$\beta+\tilde{\mathfrak{a}}_{-r}(\Lambda)\subseteq\tilde{\mathfrak{a}}'_{-r'}(\Lambda')$$
and $\frac{r'}{e(\Lambda'|\mathrm{F})}$ is smaller than $\frac{n}{e(\Lambda|\mathrm{F})}$

From now on we assume in this section that π has positive depth. By Proposition 3.6 there exists a self-dual semisimple stratum $[\Lambda,n,n-1,\beta]$ contained in π. Now let $\mathcal{M}$

and $\mathcal{M}_{\mathrm{F}}$ be the Levi sub-algebra of $\mathrm{A} = \mathrm{End}_{\mathrm{D}}\mathrm{V}$ and $\mathrm{A}_{\mathrm{F}} = \mathrm{End}_{\mathrm{F}}\mathrm{V}$, respectively, which stabilizes the associated decomposition

$$\bigoplus_{i\in\mathrm{I}} \mathrm{V}^i = \mathrm{V} \tag{3.10}$$

of β, and we denote by $\mathcal{M}_{\mathrm{L}} \subseteq \mathrm{End}_{\mathrm{L}}\mathrm{V}$ the Levi sub-algebra corresponding to $\beta\otimes 1$. Then $\mathcal{M}_{\mathrm{L}} \subseteq \mathcal{M}\otimes_{\mathrm{F}}\mathrm{L}$. We formulate the key proposition for the induction step for Theorem 3.1:

Proposition 3.11 (see [41] 5.4 over L)**.** Given a self-dual semisimple stratum Δ with positive r, an element $\theta\in\mathrm{C}(\Delta(1-))$ and an element $c\in\mathfrak{a}_{-r}\cap\mathcal{M}$, we suppose that $\theta\psi_c$ is contained in π. We fix a self-dual tame corestriction s_β with respect to β. Let Λ' be a self-dual $\mathfrak{o}_E$-$\mathfrak{o}_D$-lattice sequence, r' a positive integer and b' an element of $\mathfrak{b}'_{-r'}\cap\mathfrak{b}_{-r}$ such that $s_\beta(c)+\tilde{\mathfrak{b}}_{1-r}$ is contained in $b'+\tilde{\mathfrak{b}}'_{1-r'}$. Suppose further that $\frac{r'}{e(\Lambda'|F)}\le\frac{r}{e(\Lambda|F)}$. Then Δ' with $\beta' := \beta$ is a self-dual semisimple stratum and there are $\theta'\in\mathrm{C}(\Delta'(1-))$ and $c'\in\mathfrak{a}'_{-r'}$ such that $s_\beta(c')$ is equal to b' and $\theta'\psi_{c'}$ is contained in π. The element c' can be chosen to vanish if $b'=0$.

Essentially **Proposition 3.11** says that if $\theta\psi_c$ is contained in π then one can work in B to look for a "better" character $\theta'\psi_{c'}$. Note that assuming b' to be also contained in $\mathfrak{b}_{-r}$ is not a restriction, because we could just take $b'=s(c)$. **We explain the strategy of its proof** (taken from [41, 5.4]): At first one constructs open compact subgroups $\tilde{\mathrm{K}}_1^t(\Lambda)$ and $\tilde{\mathrm{K}}_2^t(\Lambda)$ $(t\in\mathbb{N})$ of $\tilde{\mathrm{P}}^r(\Lambda)$ via

$$\tilde{\mathrm{K}}_1^t(\Lambda) := 1+\tilde{\mathfrak{a}}_{\lfloor\frac{t}{2}\rfloor+1}\cap\Big(\prod_{i\neq j}A^{ij}\oplus\prod_i\tilde{\mathfrak{a}}_t^i\Big), \tag{3.12}$$

$$\tilde{\mathrm{K}}_2^t(\Lambda) := 1+\tilde{\mathfrak{a}}_{\lfloor\frac{t+1}{2}\rfloor}\cap\Big(\prod_{i\neq j}A^{ij}\oplus\prod_i\tilde{\mathfrak{a}}_t^i\Big), \tag{3.13}$$

and further $\tilde{\mathrm{H}}_i^t(\beta,\Lambda)$ and $\tilde{\mathrm{J}}_i^t(\beta,\Lambda)$ as intersections of $\tilde{\mathrm{H}}(\beta,\Lambda)$ and $\tilde{\mathrm{J}}(\beta,\Lambda)$ with $\tilde{\mathrm{K}}_i^t(\Lambda)$. And we get the groups $\mathrm{K}_i^t,\mathrm{H}_i^t$ and J_i^t if we intersect further down to G. We extend θ to a semisimple character of $\mathrm{H}^{\lfloor\frac{r}{2}\rfloor+1}(\beta,\Lambda)$ (which we still call θ) and we consider the character $\xi := \theta\psi_c$ on $\mathrm{H}_1^r(\beta,\Lambda)$. Mutatis mutandis as in [41, 5.7] one shows that ξ is contained in π (using **I**4.51, instead of [41, 3.17], and [41, 3.20] on $\mathrm{G}\otimes_{\mathrm{F}}\mathrm{L}$ followed by restriction to G.) This representation ξ is very helpful for detecting if a certain representation is contained in π:

Lemma 3.14 (cf. [11] (8.1.7), [41] 5.8)**.** We granted $r>0$. Let ρ be an irreducible representation on an open subgroup U of $\mathrm{K}_2^r(\Lambda)$. Then ρ is a subrepresentation of π, if its restriction to $\mathrm{U}\cap\mathrm{H}_1^r(\beta,\Lambda)$ contains $\xi|_{\mathrm{U}\cap\mathrm{H}_1^r(\beta,\Lambda)}$.

By [41, 5.12] we can cut the line in $\mathfrak{B}(\mathrm{G})$ between Λ and Λ' into segments with cutting points $\Lambda =: \Lambda_1,\Lambda_2,\dots,\Lambda_s := \Lambda'$ such that for each index $1\le k<s$ we have $\tilde{\mathrm{P}}_{r_{k+1}}(\Lambda_{k+1})\subseteq\tilde{\mathrm{K}}_2^{r_k}(\Lambda_{\mathrm{k}})$. For the definition of r_t, see [41, 5.1]. Indeed: One applies *loc.cit.* to the line in $\mathfrak{B}(\mathrm{Aut}_{\mathrm{F}}(\mathrm{V}))$ and intersects the inclusions down to $\tilde{\mathrm{G}}$. They still satisfy $\frac{r_t}{e(\Lambda_t|F)}\ge\frac{r_{t+1}}{e(\Lambda_{t+1}|F)}$. By *loc.cit.* this reduces Proposition 3.11 to the case $s=2$. Thus we have to prove Case $s=2$ and Lemma 3.14 to obtain Proposition 3.11.

For the proof of Lemma 3.14 we need:

Lemma 3.15 (cf. [41] 5.9, [21] 3.4)**.** Granted $r > 0$, there is a unique irreducible representation μ of J_1^r containing ξ (called the Heisenberg representation of ξ to J_1^r), because the bi-linear form

$$k_\xi : J_1^r/H_1^r \times J_1^r/H_1^r \to \mathbf{C}^\times,\ k_\xi(\bar{x},\bar{y}) := \xi([x,y]) = \theta([x,y])$$

is non-degenerate.

We will use base extension to L for the proof. So we use the following notation over L:

Notation 3.16. If β and Λ are fixed and $t \in \mathbb{N}_0$ then we write:

- $\tilde{H}_L^t = \tilde{H}^t(\beta \otimes 1, \Lambda_L)$, $\tilde{J}_L^t = \tilde{J}^t(\beta \otimes 1, \Lambda_L)$ and H_L^t, J_L^t for their intersections with $G \otimes_F L$ (They all are subgroups of $\tilde{G} \otimes_F L = \mathrm{Aut}_L(V)$).
- We also define, for $s \in \{1,2\}$, the groups $\tilde{K}_{L,s}^t(\Lambda)$ in replacing in (3.12) and (3.13) $\tilde{\mathfrak{a}}_*(\Lambda)$ by $\tilde{\mathfrak{a}}_*(\Lambda_L)$ and A^{ij} by $(A \otimes_F L)^{ij}$, $i,j \in I$. Note that this is not the group given in [41, 5.2], because $L[\beta \otimes 1]$ could have more factors than $F[\beta]$ and we consider $\mathcal{M} \otimes_F L$ instead of $\mathcal{M}_L$.
- We then define the groups

$$J_{L,s}^t = \tilde{K}_{L,s}^t \cap J_L^t,\ H_{L,s}^t = \tilde{K}_{L,s}^t \cap H_L^t.$$

Proof. Let ξ_L and θ_L be the unique Gal(L|F)-fixed extensions to $H_{L,1}^r$ and $H_L^{\lfloor \frac{r}{2} \rfloor + 1}$ of ξ and θ, respectively. We have the analogous form k_{ξ_L} on $J_{L,1}^r/H_{L,1}^r$ for ξ_L, and this form is non-degenerate by the proof in [41, 5.9]. Let $\bar{x}$ be in the kernel of k_ξ and let y be an element of $J_{L,1}^r$. Then

$$k_{\xi_L}(\bar{x},\bar{y}) = \theta_L([x,y]) = \theta_L([x,\tau(y)]) = k_{\xi_L}(\bar{x},\overline{\tau(y)}).$$

In particular

$$k_{\xi_L}(\bar{x},\bar{y})^2 = k_{\xi_L}(\bar{x},\bar{y}\tau(\bar{y})).$$

We take here the obvious Galois-action on $J_{L,1}^r/H_{L,1}^r$. Its fixed point set is J_1^r/H_1^r because the first Gal(L|F)-cohomology of $H_{L,1}^r$ is trivial, in particular $\bar{y}\tau(\bar{y})$ is an element of J_1^r/H_1^r. Thus $k_{\xi_L}(\bar{x},\bar{y})^2$ vanishes and therefore $k_{\xi_L}(\bar{x},\bar{y}) = 1$, because it is a p-th root of unity. Thus k_ξ is non-degenerate. □

Proof of Lemma 3.14. For the proof we skip the argument Λ in the notation. The proof needs two parts: We show

(i) There exists up to isomorphism only one irreducible representation ω of K_2^r which contains ξ. In fact we will further obtain that $\mathrm{ind}_{H_1^r}^{K_2^r}\xi$ is a multiple of ω.
(ii) The restriction of π to U contains ρ.

Part (ii) follows as in the final argument in the proof of [11, (8.1.7)]. We only have to prove Part (i). We take the representation μ of Lemma 3.15 and prove that the $\omega := \mathrm{ind}_{J_1^r}^{K_2^r}\mu$

is irreducible. Let g be an element of K_2^r which intertwines μ. Then the g-intertwining space $\mathrm{I}_g(\mu)$ satisfies the formula:

$$(\mathrm{J}_1^r:\mathrm{H}_1^r)\dim_{\mathbf{C}}(\mathrm{I}_g(\mu))=\dim_{\mathbf{C}}(\mathrm{I}_g(\mathrm{ind}_{\mathrm{H}_1^r}^{\mathrm{J}_1^r}\xi))=|\mathrm{H}_1^r\backslash\mathrm{J}_1^r g\mathrm{J}_1^r/\mathrm{H}_1^r|$$

by [11, (4.1.5)]. The latter cardinality is equal to

$$\frac{(\mathrm{J}_1^r:\mathrm{H}_1^r)(\mathrm{J}_1^r:\mathrm{J}_1^r\cap g^{-1}\mathrm{J}_1^r g)}{(\mathrm{H}_1^r:\mathrm{H}_1^r\cap g^{-1}\mathrm{H}_1^r g)},$$

and therefore odd. Thus g intertwines the Glaubeman transfer μ_{L} of μ by [39, 2.4], in particular g is an element of $\mathrm{J}_{\mathrm{L},1}^r$ by the proof of [41, 5.9]. (It shows $\mathrm{I}_{\mathrm{K}_{\mathrm{L},2}^r}(\mu_{\mathrm{L}})\subseteq\mathrm{J}_{\mathrm{L},1}^r$, see the second part of the proof in *loc.cit.* and Remark 3.17 below.) Thus $g\in\mathrm{J}_1^r$. Therefore ω is irreducible and

$$\mathrm{ind}_{\mathrm{H}_1^r}^{\mathrm{K}_2^r}\xi\cong\mathrm{ind}_{\mathrm{J}_1^r}^{\mathrm{K}_2^r}\mathrm{ind}_{\mathrm{H}_1^r}^{\mathrm{J}_1^r}\xi\cong\mathrm{ind}_{\mathrm{J}_1^r}^{\mathrm{K}_2^r}\mu^{\oplus(\mathrm{J}_1^r:\mathrm{H}_1^r)^{\frac{1}{2}}}\cong\omega^{\oplus(\mathrm{J}_1^r:\mathrm{H}_1^r)^{\frac{1}{2}}}$$

which finishes the proof. □

Remark 3.17. The proof in [41, 5.9] uses an Iwahori decomposition adapted to $\mathcal{M}_{\mathrm{L}}$. But still the proof works with an Iwahori decomposition adapted to $\mathcal{M}\otimes_{\mathrm{F}}\mathrm{L}$ which may properly contain $\mathcal{M}_{\mathrm{L}}$. At the end of the proof in *loc.cit.* the author refers to [11, (8.1.8)] for the *simple* case. In our situation, where we split a stratum over L with respect to $\mathcal{M}\otimes_{\mathrm{F}}\mathrm{L}$, the proof in [11, (8.1.8)] is applied to the *quasi-simple* case (a notion introduced by Sechérre in [29]), i.e. to a stratum $\Delta\otimes\mathrm{L}$ where Δ is a simple stratum over D. The used part of the proof in [11, (8.1.8)] is the induction.

To finally prove Proposition 3.11 we need the Cayley map (depending on Λ)

$$\mathrm{Cay}:\ \mathfrak{a}_1\to\mathrm{P}_1(\Lambda),\ \mathrm{Cay}(a):=\frac{1+\frac{a}{2}}{1-\frac{a}{2}}.$$

It is a bijection.

Proof of Proposition 3.11. We only need to consider the case $s=2$ by the above explanation. We are going to use base extension to use parts of the proof of [41, 5.4 (with assumption (H))]. We need to find $\theta'\in\mathrm{C}(\Delta'(1-))$ and $c'\in\mathfrak{a}_{-r'}(\Lambda')$ such that $\theta'\psi_{c'}\subseteq\pi$ and $s(c')=b'$.

Step 1: At first, $\Delta'=[\Lambda',n',r',\beta]$ is semisimple, because

$$\frac{r'}{e(\Lambda'|\mathrm{F})}\le\frac{r}{e(\Lambda|\mathrm{F})}<\frac{-k_0(\beta,\Lambda)}{e(\Lambda|\mathrm{F})}=\frac{-k_0(\beta,\Lambda')}{e(\Lambda'|\mathrm{F})}$$

Step 2 (find θ'): Let $\theta_{\mathrm{L}}\in\mathrm{C}((\Delta\otimes\mathrm{L})(1-))$ be the Glauberman lift of $\theta\in\mathrm{C}(\Delta(1-))$. We write Λ_{L} for the lattice sequence Λ seen as an o_{L}-lattice sequence. Note that we have

$$\mathrm{C}((\Delta\otimes\mathrm{L})(1-))=\mathrm{C}(e(\Lambda'|\mathrm{F})\Lambda_{\mathrm{L}},e(\Lambda'|\mathrm{F})r-1,\beta\otimes1)$$

and

$$e(\Lambda'|\mathrm{F})r-1\ge e(\Lambda|\mathrm{F})r'-1,$$

and we take an extension $\theta^1_{\mathrm{L}} \in \mathrm{C}(e(\Lambda'|\mathrm{F})\Lambda_{\mathrm{L}}, e(\Lambda|\mathrm{F})r'-1, \beta\otimes 1)$ of θ_{L}. Let θ'_{L} be the transfer of θ^1_{L} to

$$\mathrm{C}(e(\Lambda|\mathrm{F})\Lambda'_{\mathrm{L}}, e(\Lambda|\mathrm{F})r'-1, \beta\otimes 1) = \mathrm{C}(\Lambda'_{\mathrm{L}}, r'-1, \beta\otimes 1),$$

and we define θ' as the restriction of θ'_{L} to $\mathrm{H}^{r'}(\beta,\Lambda') = \mathrm{H}(\Delta'(1-)) = \mathrm{H}^{r'}(\beta\otimes 1, \Lambda'_{\mathrm{L}}) \cap \mathrm{G}$.

Step 3 (finding c'): Here we restrict to F and we write Λ_{F} if we consider Λ as a lattice sequence over F. At first the map $s_{\mathrm{F}} = s \otimes_{\mathrm{F}} \mathrm{id}_{\mathrm{End}_{\mathrm{A}}(\mathrm{V})}$ from $\mathrm{A}_{\mathrm{F}} = \mathrm{A}\otimes_{\mathrm{F}} \mathrm{End}_{\mathrm{A}}(\mathrm{V})$ to B_{F} is a tame corestriction relative to $\mathrm{F}[\beta]/\mathrm{F}$ by definition [35, 4.13]. By [41, 5.4 after (H)] there exists an element

$$c'_{\mathrm{F}} \in \mathfrak{a}_{-r}(\Lambda_{\mathrm{F}}) \cap \mathfrak{a}_{-r'}(\Lambda'_{\mathrm{F}}) \cap \mathcal{M}_{\mathrm{F}}$$

such that $s_{\mathrm{F}}(c'_{\mathrm{F}}) = b'$. We take the average

$$c' = \varnothing_{\langle \tau_l, \tau_{\varpi_{\mathrm{D}}}\rangle}(c'_{\mathrm{F}}) \in \mathfrak{a}_{-r}(\Lambda) \cap \mathfrak{a}_{-r'}(\Lambda') \cap \mathcal{M}.$$

Step 4 (to show $\theta'\psi_{c'} \subseteq \pi$): We still follow the proof of [41, 5.4 after (H)]. Define $\delta := c' - c$. By the proof of *loc.cit.* using [11, 8.1.13,1.4.10] there exists an element $x_{\mathrm{F}} \in \mathfrak{m}(\Delta_{\mathrm{F}}) \cap \mathcal{M}_{\mathrm{F}}$ such that

$$\delta - a_\beta(x_{\mathrm{F}}) \in \mathfrak{a}_{-r}(\Lambda_{\mathrm{F}}) \cap \mathfrak{a}_{1-r'}(\Lambda'_{\mathrm{F}}) \cap \mathcal{M}_{\mathrm{F}}.$$

Take $x := \varnothing_{\langle \tau_l, \tau_{\varpi_{\mathrm{D}}}\rangle}(x_{\mathrm{F}})$. Then x is an element of $\mathfrak{m}(\Delta)$ and therefore an element of $\mathfrak{m}(\Delta\otimes\mathrm{L})$. In particular $\mathrm{Cay}(x)$ is an element of $\mathrm{I}(\theta_{\mathrm{L}})$ and normalizes θ_{L}. Therefore by [41, 3.21] we have on $\mathrm{H}^r(\beta\otimes 1, \Lambda_{\mathrm{L}}) \cap \mathrm{H}^{r'}(\beta\otimes 1, \Lambda'_{\mathrm{L}})$:

$$\theta_{\mathrm{L}}^{\mathrm{Cay}(x)} = \theta_{\mathrm{L}}\psi_{\mathrm{Cay}(x)^{-1}\beta\mathrm{Cay}(x)-\beta} = \theta_{\mathrm{L}}\psi_{\mathrm{Cay}(x)^{-1}\delta} = \theta_{\mathrm{L}}\psi_\delta,$$

which implies that $(\theta_{\mathrm{L}}\psi_c)^{\mathrm{Cay}(x)}$ and $\theta'_{\mathrm{L}}\psi_{c'}$ coincide there. The element $\mathrm{Cay}(x)$ stabilizes $\mathrm{H}^r(\beta,\Lambda)$ (**I**5.39, because $\mathrm{Cay}(x)$ stabilizes the stratum Δ) and also $\mathrm{K}^r_2(\Lambda)$ (because $x \in \mathcal{M}\cap\mathfrak{a}_1(\Lambda)$). Fix an Iwahori decomposition with respect to (3.10). The characters ξ and $\theta'\psi_{c'}$ respect this Iwahori decomposition and are trivial on the lower and upper unipotent parts. Thus, as $\mathrm{Cay}(x) \in \prod_{i\in\mathrm{I}} \mathrm{A}^{ii}$, ξ and $(\theta'\psi_{c'})^{\mathrm{Cay}(x)^{-1}}$ coincide on the intersection of their domains. We still have

$$\mathrm{H}^{r'}(\beta,\Lambda') \subseteq \mathrm{P}_{r'}(\Lambda') \subseteq \mathrm{K}^r_2(\Lambda)$$

by assumption for the case $s=2$. We apply Lemma 3.14 to obtain

$$\theta'\psi_{c'} \subseteq \pi.$$

□

The theory of optimal points gives the following lemma.

Lemma 3.18 (cf.[40] 4.3,[26] 6.1)**.** (i) Let Λ be a lattice sequence of D-period e, and m a positive integer such that $\tilde{\mathfrak{a}}_{-m}(\Lambda) \neq \tilde{\mathfrak{a}}_{-m+1}(\Lambda)$. Then there is a lattice chain Λ' of D-period e' and an integer m' such that $\frac{m'}{e'} \leq \frac{m}{e}$ and $\tilde{\mathfrak{a}}_{-m}(\Lambda) \subseteq \tilde{\mathfrak{a}}'_{-m'}(\Lambda')$.

(ii) Let Λ be a self-dual lattice sequence of D-period e and m a positive integer such that $\tilde{\mathfrak{a}}_{-m}(\Lambda) \neq \tilde{\mathfrak{a}}_{-m+1}(\Lambda)$. Then there is a self-dual lattice sequence Λ' of D-period e' smaller than $2\dim_{\mathrm{D}}\mathrm{V}$ and an integer m' such that $\frac{m'}{e'} \leq \frac{m}{e}$ and $\tilde{\mathfrak{a}}_{-m}(\Lambda) \subseteq \tilde{\mathfrak{a}}'_{-m'}(\Lambda')$.

The main point of the lemma is that e' is bounded. The skewfield D does not play a role, i.e. the proof of 3.18(i) is the same for F and D. We give here a very simple proof of the above lemma using a different idea than roots.

Proof. The second assertion follows directly from the first by applying $()^{\#}$ and taking the barycentre. Without loss of generality we can assume that $\frac{m}{e}$ is smaller than 1. We reformulate the statement.

We consider a point x in $\mathfrak{B}(\tilde{\mathrm{G}})$ and $t \in]0, \frac{1}{d}[$. The point $[x]$ of $\mathfrak{B}_{red}(\tilde{\mathrm{G}})$ lies in the closure of a chamber C. *Then there is a midpoint $[y]$ of a facet of* C *such that $\tilde{\mathfrak{g}}_{x,-t} \subseteq \tilde{\mathfrak{g}}_{y,-t}$.*

For simplicity we assume $d = 1$, i.e. we prove the statement over F. (or one just rescales to get for $\tilde{\mathfrak{g}}_x$ the period 1.) For a lattice M which occurs in the image of a lattice function Γ corresponding to x we set s_{M} to be the maximum of all real s such that $\Gamma(s) = \mathrm{M}$. We define the following sequence of real numbers

$$s_0 := 0,\ s_j := s_{\Gamma(s_{j-1}-t)},\ j \geq 0.$$

At first we observe that the sequence gets periodic mod $\mathbb{Z}$, say the period is given by $s_{j+1}, \cdots, s_{j+e'} \equiv s_j$ mod $\mathbb{Z}$. Let $[y] \in \mathfrak{B}_{red}(\tilde{\mathrm{G}})$ be the barycentre of the facet whose vertexes correspond to the homothety classes of the lattices $\Gamma(s_{j+l})$, $l = 1, \cdots, e'$. Note that these homothety classes differ pairwise. We write u for $s_j - s_{j+e'}$, in particular $t \geq \frac{u}{e'}$, and let Γ' be a lattice function corresponding to y. Let Γ'' be the lattice function obtained from Γ in deleting all lattices from Γ which are not in the image of Γ', i.e. if $\Gamma(s)$ does not occur in the image of Γ' then we replace $\Gamma(s)$ by $\Gamma((s+v)+)$ where v is the smallest non-negative real number such that $\Gamma((s+v)+)$ is in the image of Γ'. Then, for $i \geq 0$, $\Gamma''(]s_{j+i+1}, s_{j+i}])$ contains exactly u lattices because there are no repetitions in the period. Indeed:, if $t_1, t_2 \in]s_{j+i+1}, s_{j+i}]$ satisfy $\Gamma''(t_1) \supsetneq \Gamma''(t_2)$ then $\Gamma(s_{\Gamma''(t_1)}) \supsetneq \Gamma(s_{\Gamma''(t_2)})$ and $s_{j+i+2} < s_{\Gamma(s_{\Gamma''(t_1)}-t)} < s_{\Gamma(s_{\Gamma''(t_2)}-t)} \leq s_{j+i+1}$. So we get injective maps:

$$\Gamma''(]s_{j+1}, s_j]) \hookrightarrow \Gamma''(]s_{j+2}, s_{j+1}]) \ldots \hookrightarrow \Gamma''(]s_{j+e'+1}, s_{j+e'}]),$$

in particular they are all bijections. So $\tilde{\mathfrak{g}}_{x,-t} \subseteq \tilde{\mathfrak{g}}_{y,-\frac{u}{e'}} \subseteq \tilde{\mathfrak{g}}_{y,-t}$. □

We obtain the following immediate corollary.

Corollary 3.19. We can find Λ' in Corollary 3.8(ii) and Corollary 3.9(ii) with D-period not greater than $2\dim_{\mathrm{D}} \mathrm{V}$ and $\dim_{\mathrm{D}} \mathrm{V}$, respectively.

Now we are able to finish the proof of Theorem 3.1.

Proof of Theorem 3.1. The proof is similar to the first part of the argument after the proof of [41, 5.5]. Let z be the minimal element of $\frac{1}{(4N)!}\mathbb{Z}$ ($N := \dim_{\mathrm{F}} V$) such that there is a self-dual semisimple character $\theta \in \mathrm{C}(\Delta)$ contained in π with $\frac{r}{e(\Lambda|F)} \leq z$. We claim that z vanishes. Assume for deriving a contradiction that z is positive. We extend θ to $\mathrm{C}(\Delta(1-))$ and call it again θ and there is a $c \in \mathfrak{a}_{-r}$ such that $\theta\psi_c$ is contained in π. The element c can be chosen in $\prod_i \mathrm{A}^{ii}$ by *loc.cit.* 5.2. Let s_β be a self-dual tame corestriction with respect to β. Then the multi-stratum $[\Lambda_\beta, r, r-1, s_\beta(c)]$ has to be fundamental, i.e. at least one of the strata $[\Lambda^i_\beta, r, r-1, s_{\beta_i}(c_i)]$ has to be fundamental, by the argument in the proof of *loc.cit.* 5.5 using Proposition 3.11 and Corollary 3.19 (instead of using [40, 4.3]). Note further the latter stratum being fundamental also implies that $\frac{r}{e(\Lambda|F)}$ is an element of $\frac{1}{(4N)!}\mathbb{Z}$ by [40, 2.11] (using [31, 3.11]), i.e. $\frac{r}{e(\Lambda|F)}=z$ by the choice of z. We apply Corollary 3.9 and Corollary 3.8 to choose for every $i\in I_{0,+}$ a semisimple stratum $[\Xi^i,r_i,r_i-1,\alpha_i]$ or $[\Xi^i,r_i,r_i,\alpha_i=0]$ such that

(i) the stratum is self-dual if $i\in I_0$,
(ii) $s_{\beta_i}(c_i)+\tilde{\mathfrak{a}}_{1-r}(\Lambda^i_\beta)\subseteq\alpha_i+\tilde{\mathfrak{a}}_{1-r_i}(\Xi^i)$, for all $i\in I_0\cup I_+$, and
(iii) $\frac{r}{e(\Lambda^i_\beta|E_i)}\geq\frac{r_i}{e(\Xi^i|E_i)}$, for all $i\in I_{0,+}$, with equality if $[\Lambda^i_\beta,r,r-1,s_\beta(c_i)]$ is fundamental.

We take a self-dual $\mathfrak{o}_E$-$\mathfrak{o}_D$-lattice sequence Λ' such that Λ'^i_β is in the affine class of Ξ^i for every $i\in I_{0,+}$, see **II**.5.3, and $e(\Lambda|F)|e(\Lambda'|F)$. We put $r':=\frac{e(\Lambda'|F)r}{e(\Lambda|F)}$, and we consider the multi-stratum $[\Lambda'_\beta,r',r'-1,s_\beta(c)]$. We have

$$s_\beta(c)+\tilde{\mathfrak{b}}_{1-r}(\Lambda)\subseteq s_\beta(c)+\tilde{\mathfrak{b}}_{1-r'}(\Lambda'),\ s_\beta(c)\in\tilde{\mathfrak{b}}_{-r}(\Lambda)\cap\tilde{\mathfrak{b}}_{-r'}(\Lambda').$$

Thus, by Proposition 3.11, there is a self-dual semisimple stratum Δ' with $\beta'=\beta$ and a character $\theta'\in C(\Delta'(1-))$ and an element $c'\in\mathfrak{a}'_{-r'}$ such that $\theta'\psi_{c'}$ is contained in π and $s_\beta(c')=s_\beta(c)$. Now Δ' is semisimple and $[\Lambda_\beta,r',r'-1,s_\beta(c)]$ is equivalent to a semisimple multi-stratum. Then $[\Lambda',n',r'-1,\beta'+c']$ is equivalent to a semisimple stratum by **I**.4.15. Further the self-duality of the stratum implies that it is equivalent to a self-dual semisimple stratum, by **II**.4.7, say $\Delta''=[\Lambda',n',r''=r'-1,\beta'']$. Then $C(\Delta'')=C(\Delta'(1-))\psi_{c'}$. Thus there is an element θ'' of $C(\Delta'')$ contained in π and

$$\frac{r''}{e(\Lambda''|F)}=\frac{r'-1}{e(\Lambda'|F)}<\frac{r'}{e(\Lambda'|F)}=\frac{r}{e(\Lambda|F)}=z.$$

Note that on the other hand we could have started with θ'' and therefore $\frac{r''}{e(\Lambda''|F)}=z$. A contradiction. □

4. Heisenberg representations

The study of Heisenberg representations and their extensions are the technical heart of Bushnell–Kutzko theory, for both: the construction of cuspidal representations and the exhaustion. We will review the results for $G_L:=G\otimes L$ and extend them to G. In this section we fix a stratum $\Delta=[\Lambda,n,0,\beta]$. Let Λ' be an $\mathfrak{o}_E$-$\mathfrak{o}_D$-lattice sequence which satisfies $\tilde{\mathfrak{b}}(\Lambda')\subseteq\tilde{\mathfrak{b}}(\Lambda)$. Let us recall that we have the following sequence of groups:

$$H^i_\Lambda:=H^i(\beta,\Lambda),\ J^i_\Lambda:=J^i(\beta,\Lambda),\ i\in\mathbb{N},\ J_\Lambda:=J(\beta,\Lambda),J^0_\Lambda:=J^1(\beta,\Lambda)P^0(\Lambda_\beta)$$

and

$$J^1_{\Lambda',\Lambda}:=J^1_\Lambda P_1(\Lambda'_\beta),\ J^0_{\Lambda',\Lambda}:=J^1_\Lambda P^0(\Lambda'_\beta),\ J_{\Lambda',\Lambda}:=J^1_\Lambda P(\Lambda'_\beta).$$

We have similar subgroups $J^i_{\Lambda'_L,\Lambda_L}$ and $H^i_{\Lambda_L}$ of G_L. We fix a character $\theta\in C(\Delta)$, and let θ' be the transfer of θ from Λ to Λ'. We denote the Gal(L|F)-Glauberman lifts of θ and θ' by θ_L and θ'_L.

At first we recall the Heisenberg representations for G_L.

Proposition 4.1 ([41] 3.29,3.31,[21] 5.1)**.** (i) There is up to equivalence a unique irreducible representation $(\eta_{\Lambda_{\mathrm{L}}}, \mathrm{J}^1_{\Lambda_{\mathrm{L}}})$ which contains θ_{L}.
(ii) Let g be an element of $\mathrm{G_L}$. The $\mathbf{C}$-dimension of $\mathrm{I}_g(\eta_{\Lambda_{\mathrm{L}}}, \eta_{\Lambda'_{\mathrm{L}}})$ is at most one, and it is one if and only if $g \in \mathrm{J}^1_{\Lambda'_{\mathrm{L}}}(\mathrm{G_L})_\beta \mathrm{J}^1_{\Lambda_{\mathrm{L}}}$.

We want to prove its analogue for G. At first we need a lemma which allows us to apply Bushnell–Fröhlichs' work to construct Heisenberg representations:

Lemma 4.2 (cf. [41] 3.28 for $\mathrm{G_L}$.)**.** The form

$$\mathrm{k}_\theta : \mathrm{J}^1_\Lambda/\mathrm{H}^1_\Lambda \times \mathrm{J}^1_\Lambda/\mathrm{H}^1_\Lambda \to \mathbf{C}$$

defined via $\mathrm{k}_\theta(\bar{x}, \bar{y}) := \theta([x, y])$ is non-degenerate. The pair $(\mathrm{J}^1_\Lambda/\mathrm{H}^1_\Lambda, \mathrm{k}_\theta)$ is a subspace of $(\mathrm{J}^1_{\Lambda_{\mathrm{L}}}/\mathrm{H}^1_{\Lambda_{\mathrm{L}}}, \mathrm{k}_{\theta_{\mathrm{L}}})$ ($\mathrm{k}_{\theta_{\mathrm{L}}}$ similarly defined).

The proof is similar to the proof of Lemma 3.15.

Proposition 4.3. (i) There is up to equivalence a unique irreducible representation η_Λ of J^1_Λ which contains θ. Further η_Λ has degree $(\mathrm{J}^1_\Lambda : \mathrm{H}^1_\Lambda)^{\frac{1}{2}}$.
(ii) The representation η_Λ is the Gal(L|F)-Glauberman transfer of $\eta_{\Lambda_{\mathrm{L}}}$ to J^1_Λ.
(iii) Let g be an element of G. The $\mathbf{C}$-dimension of $\mathrm{I}_g(\eta_\Lambda, \eta_{\Lambda'})$ is at most one, and it is one if and only if $g \in \mathrm{J}^1_{\Lambda'}\mathrm{G}_\beta \mathrm{J}^1_\Lambda$.

Proof. We define η_Λ as $\mathbf{gl}^{\mathrm{L|F}}_{\mathbf{C}}(\eta_{\Lambda_{\mathrm{L}}})$, see Proposition 4.1. The restriction of $\eta_{\Lambda_{\mathrm{L}}}$ to H^1_Λ is a multiple of θ. Thus the same is true for η_Λ. And thus by [10, 8.1] and Lemma 4.2 up to equivalence η_Λ is the unique irreducible representation of J^1_Λ which contains θ, and further it has the desired degree. An element of G_β intertwines $\eta_{\Lambda_{\mathrm{L}}}$ with $\eta_{\Lambda'_{\mathrm{L}}}$ with intertwining space of dimension 1, so it intertwines η_Λ with $\eta_{\Lambda'}$, by [39, 2.4]. On the other hand we have

$$\mathrm{I_G}(\eta_\Lambda, \eta_{\Lambda'}) \subseteq \mathrm{I_G}(\theta, \theta') = \mathrm{J}^1_{\Lambda'}\mathrm{G}_\beta \mathrm{J}^1_\Lambda,$$

which finishes the proof of the intertwining formula. The most complicated part is the proof of dimension one of the non-zero intertwining spaces. For this we refer to the proof of [21, 5.1]. Note that after taking Gal(L|F)-fixed points in the rectangular diagram of [21, 5.2] the rows and columns remain still exact by the additive Hilbert 90. The rest of the proof is mutatis mutandis.

□

For the exhaustion the following extensions of η_Λ are the key technical tools. We will emphasize the importance when their application arises. Note that we say that a representation $(\tilde{\gamma}, \tilde{\mathrm{K}})$ is an *extension* of a representation (γ, K) if K is a subgroup of $\tilde{\mathrm{K}}$ and the restriction of $\tilde{\gamma}$ to K is equivalent to γ.

Proposition 4.4 ([42] 3.7, [21] 5.6, 5.7)**.** Suppose $\tilde{\mathfrak{a}}(\Lambda') \subseteq \tilde{\mathfrak{a}}(\Lambda)$. There is up to equivalence a unique irreducible representation $(\eta_{\Lambda'_{\mathrm{L}},\Lambda_{\mathrm{L}}}, \mathrm{J}^1_{\Lambda'_{\mathrm{L}},\Lambda_{\mathrm{L}}})$ which extends $(\eta_{\Lambda_{\mathrm{L}}}, \mathrm{J}^1_{\Lambda_{\mathrm{L}}})$ such that $\eta_{\Lambda'_{\mathrm{L}},\Lambda_{\mathrm{L}}}$ and $\eta_{\Lambda'_{\mathrm{L}}}$ induce equivalent irreducible representations on $\mathrm{P}_1(\Lambda'_{\mathrm{L}})$. Moreover the set of intertwining elements of $\eta_{\Lambda'_{\mathrm{L}},\Lambda_{\mathrm{L}}}$ in $\mathrm{G_L}$ is $\mathrm{J}^1_{\Lambda'_{\mathrm{L}},\Lambda_{\mathrm{L}}}(\mathrm{G_L})_\beta \mathrm{J}^1_{\Lambda'_{\mathrm{L}},\Lambda_{\mathrm{L}}}$. The intertwining spaces $\mathrm{I}_g(\eta_{\Lambda'_{\mathrm{L}},\Lambda_{\mathrm{L}}})$ have all $\mathbf{C}$-dimension at most one.

Proposition 4.5. Suppose $\tilde{\mathfrak{a}}(\Lambda') \subseteq \tilde{\mathfrak{a}}(\Lambda)$. There is up to equivalence a unique irreducible representation $(\eta_{\Lambda',\Lambda}, J^1_{\Lambda',\Lambda})$ which extends $(\eta_\Lambda, J^1_\Lambda)$ such that $\eta_{\Lambda',\Lambda}$ and $\eta_{\Lambda'}$ induce equivalent irreducible representations on $P_1(\Lambda')$. Moreover $\eta_{\Lambda',\Lambda}$ is the Gal(L|F)-Glauberman transfer of $\eta_{\Lambda'_L,\Lambda_L}$ to $J^1_{\Lambda',\Lambda}$ and the set of intertwining elements of $\eta_{\Lambda',\Lambda}$ in G is $J^1_{\Lambda',\Lambda}G_\beta J^1_{\Lambda',\Lambda}$. The intertwining spaces $I_g(\eta_{\Lambda',\Lambda})$ have all **C**-dimension at most one.

Proof. We set $\eta_{\Lambda',\Lambda}$ to be $\mathbf{gl}_{\mathbf{C}}(\eta_{\Lambda'_L,\Lambda_L})$. Now $\eta_{\Lambda',\Lambda}$ is the only irreducible representation of $J^1_{\Lambda',\Lambda}$ with an odd multiplicity in $\eta_{\Lambda'_L,\Lambda_L}$ and therefore the irreducible constituents of $\eta_{\Lambda'_L,\Lambda_L}|_{J^1_\Lambda}$ with odd multiplicity are contained in $\eta_{\Lambda',\Lambda}$, i.e. η_Λ is contained in $\eta_{\Lambda',\Lambda}$, knowing that the restriction of $\eta_{\Lambda'_L,\Lambda_L}$ to $J^1_{\Lambda_L}$ is equivalent to η_{Λ_L}. The trace condition [18, (6)] implies for the characteristic zero case that the Glauberman transfers $\mathbf{gl}^{L|F}_{\mathbb{C}}(\eta_{\Lambda'_L,\Lambda_L})$ and $\mathbf{gl}^{L|F}_{\mathbb{C}}(\eta_{\Lambda_L})$ have the same degree. We have

$$\mathrm{Br}_{J^1_{\Lambda'_L,\Lambda_L}}(\eta_{\Lambda'_L,\Lambda_L,\mathbf{C}}) = \eta_{\Lambda'_L,\Lambda_L,\mathbb{C}} \text{ and } \mathrm{Br}_{J^1_{\Lambda'_L,\Lambda_L}}(\eta_{\Lambda'_L,\mathbf{C}}) = \eta_{\Lambda'_L,\mathbb{C}}, \text{ see §2.}$$

Therefore, by Theorem 2.14, we have the equality of degrees of the Glauberman transfers in the modular case. In any case $\eta_{\Lambda',\Lambda}$ is an extension of η_Λ. As in the proof of Proposition 4.3 we obtain the formula for $I_G(\eta_{\Lambda',\Lambda})$ using Proposition 4.4 instead of Proposition 4.1. It remains to show the following three statements:

(i) The representations $\pi := \mathrm{ind}_{J^1_{\Lambda',\Lambda}}^{P_1(\Lambda')} \eta_{\Lambda',\Lambda}$ and $\mathrm{ind}_{J^1_{\Lambda'}}^{P_1(\Lambda')} \eta_{\Lambda'}$ are
   (a) irreducible, and
   (b) equivalent.
(ii) The multiplicity of η_Λ in π is one.
(iii) The intertwining spaces of $\eta_{\Lambda',\Lambda}$ have at most **C**-dimension one.

The irreducibility follows from $I_{P_1(\Lambda')}(\eta_{\Lambda',\Lambda}) = J^1_{\Lambda',\Lambda}$ and $I_{P_1(\Lambda')}(\eta_{\Lambda'}) = J^1_{\Lambda'}$. The statement about the intertwining spaces follows from Proposition 4.3. For the equivalence note at first that $\eta_{\Lambda'_L,\Lambda_L}$ has multiplicity one in

$$\mathrm{ind}_{J^1_{\Lambda'_L,\Lambda_L}}^{P_1(\Lambda'_L)} \eta_{\Lambda'_L,\Lambda_L}$$

(The latter is irreducible, so apply second Frobenius reciprocity and Schur!), in particular $\eta_{\Lambda'_L,\Lambda_L}$ has odd multiplicity in $\mathrm{ind}_{J^1_{\Lambda'_L}}^{P_1(\Lambda'_L)} \eta_{\Lambda'_L}$. Thus

$$\mathrm{Res}^{P_1(\Lambda')}_{J^1_{\Lambda',\Lambda}}(\mathbf{gl}^{L|F}_{\mathbf{C}}(\mathrm{ind}_{J^1_{\Lambda'_L}}^{P_1(\Lambda'_L)} \eta_{\Lambda'_L})) \supseteq \mathbf{gl}^{L|F}_{\mathbf{C}}(\eta_{\Lambda'_L,\Lambda_L}) = \eta_{\Lambda',\Lambda}.$$

By irreducibility $\mathrm{ind}_{J^1_{\Lambda',\Lambda}}^{P_1(\Lambda')} \eta_{\Lambda',\Lambda}$ is equivalent to $\mathbf{gl}^{L|F}_{\mathbf{C}}(\mathrm{ind}_{J^1_{\Lambda'_L}}^{P_1(\Lambda'_L)} \eta_{\Lambda'_L})$. Similarly, we obtain that $\mathbf{gl}^{L|F}_{\mathbf{C}}(\mathrm{ind}_{J^1_{\Lambda'_L}}^{P_1(\Lambda'_L)} \eta_{\Lambda'_L})$ is also equivalent to $\mathrm{ind}_{J^1_{\Lambda'}}^{P_1(\Lambda')} \eta_{\Lambda'}$. It remains to show the multiplicity assertion: Note that the set

$$\mathrm{Hom}_{J^1_\Lambda \cap {}^g J^1_{\Lambda',\Lambda}}(\eta_\Lambda, \eta^g_{\Lambda',\Lambda})$$

is trivial if $g \notin \mathrm{I_G}(\eta_\Lambda)$. Thus by Frobenius reciprocity and Mackey theory we have

$$\mathrm{Hom}_{\mathrm{J}^1_\Lambda}(\eta_\Lambda, \mathrm{ind}_{\mathrm{J}^1_{\Lambda',\Lambda}}^{\mathrm{P}_1(\Lambda')}\eta_{\Lambda',\Lambda}) = \mathrm{Hom}_{\mathrm{J}^1_\Lambda}(\eta_\Lambda, \eta_\Lambda) = \mathbf{C}.$$

This finishes the proof. □

Remark 4.6. The proof also shows that $\mathbf{gl}_{\mathbf{C}}^{\mathrm{L|F}}$ maps the class of $\eta_{\Lambda'_{\mathrm{L}},\Lambda_{\mathrm{L}}}$ to the class of $\eta_{\Lambda',\Lambda}$.

We need to show that the definition of $\eta_{\Lambda',\Lambda}$ only depends on $\tilde{\mathfrak{b}}(\Lambda')$ instead of Λ'.

Proposition 4.7. Let Λ'' be a self-dual $\mathfrak{o}_{\mathrm{E}}$-$\mathfrak{o}_{\mathrm{D}}$-lattice sequence such that $\tilde{\mathfrak{b}}(\Lambda'') = \tilde{\mathfrak{b}}(\Lambda')$ and $\tilde{\mathfrak{a}}(\Lambda'') \subseteq \tilde{\mathfrak{a}}(\Lambda)$ and suppose $\tilde{\mathfrak{a}}(\Lambda') \subseteq \tilde{\mathfrak{a}}(\Lambda)$. Then $\mathrm{J}^1_{\Lambda',\Lambda} = \mathrm{J}^1_{\Lambda'',\Lambda}$ and $\eta_{\Lambda'',\Lambda}$ is equivalent to $\eta_{\Lambda',\Lambda}$.

Proof. We consider a path of self dual $\mathfrak{o}_{\mathrm{D}}$-$\mathfrak{o}_{\mathrm{E}}$ lattice sequences $\Lambda' = \Lambda_0, \Lambda_1, \ldots, \Lambda_l = \Lambda''$ on a segment from Λ' to Λ'' in $\mathfrak{B}(\mathrm{G})$, such that

$$\tilde{\mathfrak{a}}(\Lambda_i) \cap \tilde{\mathfrak{a}}(\Lambda_{i+1}) \in \{\tilde{\mathfrak{a}}(\Lambda_i), \tilde{\mathfrak{a}}(\Lambda_{i+1})\},$$

for all $i = 0, 1, \ldots, l-1$, in particular we have $\tilde{\mathfrak{b}}(\Lambda_i) = \tilde{\mathfrak{b}}(\Lambda')$ for all $i = 0, \ldots, l$. Thus by transitivity it is enough to consider the case $\tilde{\mathfrak{a}}(\Lambda') \supseteq \tilde{\mathfrak{a}}(\Lambda'')$. The representations $\mathrm{ind}_{\mathrm{J}^1_{\Lambda',\Lambda}}^{\mathrm{P}_1(\Lambda')}\eta_{\Lambda',\Lambda}$ and $\mathrm{ind}_{\mathrm{J}^1_{\Lambda'}}^{\mathrm{P}_1(\Lambda')}\eta_{\Lambda'}$ are equivalent, and thus $\mathrm{ind}_{\mathrm{J}^1_{\Lambda',\Lambda}}^{\mathrm{P}_1(\Lambda'')}\eta_{\Lambda',\Lambda}$ is equivalent to $\mathrm{ind}_{\mathrm{J}^1_{\Lambda'}}^{\mathrm{P}_1(\Lambda'')}\eta_{\Lambda'}$. Now $\mathrm{J}^1_{\Lambda',\Lambda} = \mathrm{J}^1_{\Lambda'',\Lambda}$, $\mathrm{J}^1_{\Lambda'} = \mathrm{J}^1_{\Lambda'',\Lambda'}$, $\eta_{\Lambda'',\Lambda'} = \eta_{\Lambda'}$ and

$$\mathrm{ind}_{\mathrm{J}^1_{\Lambda'}}^{\mathrm{P}_1(\Lambda'')}\eta_{\Lambda'} \cong \mathrm{ind}_{\mathrm{J}^1_{\Lambda''}}^{\mathrm{P}_1(\Lambda'')}\eta_{\Lambda''}$$

by definition of $\eta_{\Lambda'',\Lambda'}$. Thus $\eta_{\Lambda',\Lambda}$ and $\eta_{\Lambda'',\Lambda}$ are equivalent by Proposition 4.5. □

By last proposition we can now define $\eta_{\Lambda',\Lambda}$ without assuming $\tilde{\mathfrak{a}}(\Lambda') \subseteq \tilde{\mathfrak{a}}(\Lambda)$.

Definition 4.8. Granted $\tilde{\mathfrak{b}}(\Lambda') \subseteq \tilde{\mathfrak{b}}(\Lambda)$, we define $(\eta_{\Lambda',\Lambda}, \mathrm{J}^1_{\Lambda',\Lambda})$ as the representation $(\eta_{\Lambda'',\Lambda}, \mathrm{J}^1_{\Lambda'',\Lambda})$, where Λ'' is a self-dual $\mathfrak{o}_{\mathrm{E}}$-$\mathfrak{o}_{\mathrm{D}}$-lattice sequence such that $\tilde{\mathfrak{b}}(\Lambda') = \tilde{\mathfrak{b}}(\Lambda'')$ and $\tilde{\mathfrak{a}}(\Lambda'') \subseteq \tilde{\mathfrak{a}}(\Lambda)$.

Corollary 4.9 (cf. [42] 3.8)**.** Let Λ'' be a further self-dual $\mathfrak{o}_{\mathrm{E}}$-$\mathfrak{o}_{\mathrm{D}}$-lattice sequence such that $\tilde{\mathfrak{b}}(\Lambda'') \subseteq \tilde{\mathfrak{b}}(\Lambda')$. Then the restriction of $\eta_{\Lambda'',\Lambda}$ to $\mathrm{J}^1_{\Lambda',\Lambda}$ is equivalent to $\eta_{\Lambda',\Lambda}$.

Proof. It suffices to consider the complex case, by (2.13) and Theorem 2.14. So assume $\mathbf{C} = \mathbb{C}$. The result now follows from [42, Proposition 3.8] and the Glauberman correspondence, indeed

$$\eta_{\Lambda''_{\mathrm{L}},\Lambda_{\mathrm{L}}}|_{\mathrm{J}^1_{\Lambda'_{\mathrm{L}},\Lambda_{\mathrm{L}}}} \cong \eta_{\Lambda'_{\mathrm{L}},\Lambda_{\mathrm{L}}},$$

by [42, 3.8] and $\mathbf{gl}_{\mathbb{C}}^{\mathrm{L|F}}(\eta_{\Lambda'_{\mathrm{L}},\Lambda_{\mathrm{L}}}) = \eta_{\Lambda',\Lambda}$, and thus the latter representation occurs with odd multiplicity in $\eta_{\Lambda''_{\mathrm{L}},\Lambda_{\mathrm{L}}}|_{\mathrm{J}^1_{\Lambda',\Lambda}}$, thus by Remark 4.6 $\eta_{\Lambda'',\Lambda}$ contains $\eta_{\Lambda',\Lambda}$ and hence, as they share the degree with η_Λ, we get the result. □

5. The isotypic components

In general proofs on smooth complex representations of locally totally disconnected groups cannot be easily transferred to the modular case. One trick used for the case of cuspidal representations of $\mathrm{G}\otimes\mathrm{L}$ in [21] is the following lemma:

Lemma 5.1. Let H be locally compact and totally disconnected topological group and K,K^1 be compact open subgroups of H such that K^1 is a normal pro-p subgroup of K. Let further π be a smooth $\mathbf{C}$-representation of H, and η be a smooth $\mathbf{C}$-representation of K^1 such that η is normalized by K. Then we have

$$\pi|_{\mathrm{K}} = \pi^{\eta}\oplus\pi(\eta)$$

(a direct sum of K-subrepresentations), where π^{η} is the η-isotypic component of π and $\pi(\eta)$ the largest subrepresentation of $\pi|_{\mathrm{K}^1}$ which does not contain a copy of η.

This lemma, which is trivial using the fact that $\pi|_{\mathrm{K}^1}$ is semisimple, is still very useful.

6. β-extension

In this section we generalize β-extensions to G, see [42, §4] for the case of $\mathrm{G}\otimes\mathrm{L}$ (see also [30], [11, 5.2.1] for $\tilde{\mathrm{G}}$). Its construction for classical groups is a complicated process. We fix a self-dual semisimple stratum $[\Lambda,n,0,\beta]$ and a self-dual semisimple character $\theta\in\mathrm{C}(\Lambda,0,\beta)$. We fix self-dual $\mathfrak{o}_{\mathrm{E}}$-$\mathfrak{o}_{\mathrm{D}}$ lattice sequences $\Lambda_m,\Lambda,\Lambda',\Lambda_{\mathrm{M}}$ and Λ'' such that $\tilde{\mathfrak{b}}(\Lambda_m)$ (resp. $\tilde{\mathfrak{b}}(\Lambda_{\mathrm{M}})$) is minimal (resp. maximal) and

$$\tilde{\mathfrak{b}}(\Lambda_m)\subseteq\tilde{\mathfrak{b}}(\Lambda)\subseteq\tilde{\mathfrak{b}}(\Lambda')\subseteq\tilde{\mathfrak{b}}(\Lambda_{\mathrm{M}}),$$

and such that $\tilde{\mathfrak{b}}(\Lambda)\subseteq\tilde{\mathfrak{b}}(\Lambda'')$. So we have

$$\tilde{\mathfrak{b}}(\Lambda)\subseteq\tilde{\mathfrak{b}}(\Lambda')\cap\tilde{\mathfrak{b}}(\Lambda'').$$

We are going to use the representations η_{Λ}, $\eta_{\Lambda'}$, $\eta_{\Lambda,\Lambda'}$, etc. constructed in §4. Recall that $\eta_{\Lambda'}$ is the Heisenberg representation of the transfer of θ to $C(\Lambda',0,\beta)$.

6.1. **General idea.** We present here Stevens' strategy from [42, §4]: We put

$$\mathrm{ext}(\Lambda,\Lambda') := \{(\kappa'_{\cong},\mathrm{J}_{\Lambda,\Lambda'})\in\mathfrak{R}_{\mathbf{C}}(\mathrm{J}_{\Lambda,\Lambda'})|\ \kappa'|_{\mathrm{J}^1_{\Lambda,\Lambda'}}\cong\eta_{\Lambda,\Lambda'}\},\ \mathrm{ext}(\Lambda') := \mathrm{ext}(\Lambda',\Lambda'),$$

where the subscript $\cong$ indicates the isomorphism class of the representation in question. Depending on Λ_{M} we only choose certain extensions of $\eta_{\Lambda'}$ to $\mathrm{J}_{\Lambda'}$, i.e. a subset $\beta\text{-ext}_{\Lambda_{\mathrm{M}}}(\Lambda')$ of $\mathrm{ext}(\Lambda')$, and call them β-extensions with respect to Λ_{M}.

(i) For Λ_{M} the set $\beta\text{-ext}(\Lambda_{\mathrm{M}})$ is defined to consist of those elements of $\mathrm{ext}(\Lambda_{\mathrm{M}})$ which are mapped into $\mathrm{ext}(\Lambda_m,\Lambda_{\mathrm{M}})$ under restriction to $\mathrm{J}_{\Lambda_m,\Lambda_{\mathrm{M}}}$.

(ii) For Λ' we construct a bijection Ψ from $\mathrm{ext}(\Lambda',\Lambda_{\mathrm{M}})$ to $\mathrm{ext}(\Lambda')$, see below, and the set $\beta\text{-ext}_{\Lambda_{\mathrm{M}}}(\Lambda')$ is then defined to be the image of the composition

$$\beta\text{-ext}(\Lambda_{\mathrm{M}})\overset{\mathrm{Res}^{\mathrm{J}_{\Lambda_{\mathrm{M}}}}_{\mathrm{J}_{\Lambda',\Lambda_{\mathrm{M}}}}}{\longrightarrow}\mathrm{ext}(\Lambda',\Lambda_{\mathrm{M}})\overset{\Psi}{\underset{\sim}{\longrightarrow}}\mathrm{ext}_{\Lambda_{\mathrm{M}}}(\Lambda').$$

We define a map

$$\Psi_{\Lambda,\Lambda',\Lambda''}: \ \mathrm{ext}(\Lambda,\Lambda') \to \mathrm{ext}(\Lambda,\Lambda'')$$

as follows (to get Ψ in (ii) substitute $(\Lambda,\Lambda',\Lambda'')$ by $(\Lambda',\Lambda_{\mathrm{M}},\Lambda')$):

- Consider a path of self-dual $\mathfrak{o}_{\mathrm{E}}$-$\mathfrak{o}_{\mathrm{D}}$ lattice sequences

(6.1) $$\Lambda' = \Lambda_0, \Lambda_1, \cdots, \Lambda_l = \Lambda''$$

  such that

(6.2) $$\tilde{\mathfrak{a}}(\Lambda_i)\cap\tilde{\mathfrak{a}}(\Lambda_{i+1}) \in \{\tilde{\mathfrak{a}}(\Lambda_i),\tilde{\mathfrak{a}}(\Lambda_{i+1})\},\ \tilde{\mathfrak{b}}(\Lambda)\subseteq\tilde{\mathfrak{b}}(\Lambda_i)\cap\tilde{\mathfrak{b}}(\Lambda_{i+1}).$$

  for all indexes $i\in\{0,\cdots,l-1\}$.
- Define the maps $\Psi_{\Lambda,\Lambda_i,\Lambda_{i+1}}$ in requiring isomorphic inductions to $\mathrm{P}(\Lambda_\beta)\mathrm{P}_1(\Lambda)$,
- and then put

(6.3) $$\Psi_{\Lambda,\Lambda',\Lambda''} := \Psi_{\Lambda,\Lambda_{l-1},\Lambda_l}\circ\Psi_{\Lambda,\Lambda_{l-2},\Lambda_{l-1}}\circ\cdots\circ\Psi_{\Lambda,\Lambda_0,\Lambda_1}.$$

We now give the details to those steps, beginning with the maximal case.

6.2. **The existence of β-extensions for the maximal compact case.** We are interested in extensions of η_Λ to J_Λ, but not all of them (cf. [42, Remark 4.2]). At first we define β-extensions for Λ_{M}. We define β-ext(Λ_{M}) as in §6.1(i), i.e. as the set of all isomorphism classes of irreducible representations κ of $\mathrm{J}_{\Lambda_{\mathrm{M}}}$ such that the restriction of κ to $\mathrm{J}^1_{\Lambda_m,\Lambda_{\mathrm{M}}}$ is isomorphic to $\eta_{\Lambda_m,\Lambda_{\mathrm{M}}}$.

The following proposition shows that β-ext(Λ_{M}) is non-empty. Note that $\mathrm{J}^1_{\Lambda_m,\Lambda}$ is a pro-p-Sylow subgroup of J_Λ and every pro-p-Sylow subgroup of J_Λ is of such a form, i.e. for an appropriate Λ_m, and they are all conjugate in J_Λ.

Proposition 6.4. Granted $\tilde{\mathfrak{b}}(\Lambda_m)\subseteq\tilde{\mathfrak{b}}(\Lambda)\subseteq\tilde{\mathfrak{b}}(\Lambda')$:

(i) There exists an extension $(\kappa,\mathrm{J}_\Lambda)$ of $(\eta_{\Lambda_m,\Lambda},\mathrm{J}^1_{\Lambda_m,\Lambda})$.
(ii) Let $(\kappa',\mathrm{J}_{\Lambda'})$ be an extension of $(\eta_{\Lambda_m,\Lambda'},\mathrm{J}^1_{\Lambda_m,\Lambda'})$. Then the restriction of κ' to $\mathrm{J}^1_{\Lambda,\Lambda'}$ is equivalent to $(\eta_{\Lambda,\Lambda'},\mathrm{J}^1_{\Lambda,\Lambda'})$.
(iii) Let $(\kappa,\mathrm{J}_{\Lambda_{\mathrm{M}}})$ be an extension of $(\eta_{\Lambda_{\mathrm{M}}},\mathrm{J}^1_{\Lambda_{\mathrm{M}}})$. Then are equivalent:
  (a) $\kappa\in\beta$-ext(Λ_{M}).
  (b) κ is an extension of $\eta_{\Lambda_{\mathrm{M}}}$ such that for every pro-p-Sylow subgroup S of $\mathrm{J}_{\Lambda_{\mathrm{M}}}$ the restriction of κ to S is intertwined by the whole of G_β.

For the proof we need a lemma:

Lemma 6.5 ([11] (5.3.2)(proof), cf. [21] 2.7)**.** Let K be a totally disconnected and locally compact group and let (ρ_i,W_i), $i=1,2$, be two smooth representations of K. Suppose K_2 is a normal open subgroup of K contained in the kernel of ρ_2. Suppose that the sets $\mathrm{End}_{\mathrm{K}}(\mathrm{W}_1)$ and $\mathrm{End}_{\mathrm{K}_2}(\mathrm{W}_1)$ coincide. Then:

$$\mathrm{End}_{\mathrm{K}}(\mathrm{W}_1\otimes_{\mathbf{C}}\mathrm{W}_2)\cong\mathrm{End}_{\mathrm{K}}(\mathrm{W}_1)\otimes_{\mathbf{C}}\mathrm{End}_{\mathrm{K}}(W_2).$$

In particular if K is compact and K_2 is pro-finite with $p_{\mathbf{C}}$ not dividing its pro-order we get:

(i) $\mathrm{W}_1 \otimes_{\mathbf{C}} \mathrm{W}_2$ is irreducible if and only if W_1 and W_2 are irreducible.
(ii) Suppose ρ_1 is irreducible and let ρ be an irreducible representation of K such that $\rho|_{\mathrm{K}_2}$ is isomorphic to a direct sum of copies of $\rho_1|_{\mathrm{K}_2}$. Then there is an irreducible representation ρ_2' on K containing K_2 in its kernel such that ρ is equivalent to $\rho_1 \otimes \rho_2'$.

Proof. Take a $\mathbf{C}$-basis f_i of $\mathrm{End}_{\mathbf{C}}(\mathrm{W}_2)$. We have the K-action on $\mathrm{End}_{\mathbf{C}}(\mathrm{W}_1 \otimes \mathrm{W}_2)$ via conjugation: $k.\Phi := k \circ \Phi \circ k^{-1}$, where we consider on $\mathrm{W}_1 \otimes \mathrm{W}_2$ the diagonal action of K. Then every element $\Phi = \sum_i g_i \otimes f_i$ of $\mathrm{End}_{\mathrm{K}}(\mathrm{W}_1 \otimes_{\mathbf{C}} \mathrm{W}_2)$ is fixed by K_2 and therefore g_i has to be K_2-equivariant and therefore K-equivariant by assumption. So Φ is an element of $\mathrm{End}_{\mathrm{K}}(\mathrm{W}_1 \otimes_{\mathbf{C}} \mathrm{W}_2) \cap \mathrm{End}_{\mathrm{K}}(\mathrm{W}_1) \otimes_{\mathbf{C}} \mathrm{End}_{\mathbf{C}}(W_2)$. Now a similar argument for Φ using a $\mathbf{C}$-basis of $\mathrm{End}_{\mathrm{K}}(W_1)$ shows that Φ is an element of $\mathrm{End}_{\mathrm{K}}(\mathrm{W}_1) \otimes_{\mathbf{C}} \mathrm{End}_{\mathrm{K}}(W_2)$. Now *(ii)* follows from (i) because, by (i), $\mathrm{W}_1 \otimes \mathrm{ind}_{K_2}^{\mathrm{K}} 1$ has a Jordan–Hölder composition series where all the factors are of the form $\mathrm{W}_1 \otimes_{\mathbf{C}} \mathrm{W}_2$, W_2 depending on the factor. So it remains to show (i). So, suppose W_1 and W_2 are irreducible $\mathbf{C}$-representations of K. Then, by

$$\mathrm{End}_{\mathrm{K}_2}(\mathrm{W}_1) = \mathrm{End}_{\mathrm{K}}(\mathrm{W}_1) = \mathbf{C}$$

and the pro-finiteness of K_2 with $p_{\mathbf{C}}$ not dividing the pro-order, $\mathrm{W}_1|_{\mathrm{K}_2}$ is irreducible too. Let $\tilde{\mathrm{W}}$ be a non-zero subrepresentation of $\mathrm{W}_1 \otimes_{\mathbf{C}} \mathrm{W}_2$ and

$$\sum_{i=1}^{u} w_i^{(1)} \otimes_{\mathbf{C}} w_i^{(2)}$$

a non-zero sum of elementary tensors contained in $\tilde{\mathrm{W}}$. Let u be minimal. For every pair (i_1, i_2) of indexes and any finite tuple $(k_j)_j$ of K_2 we have

$$\sum_j k_j w_{i_1}^{(1)} = 0 \text{ if and only if } \sum_j k_j w_{i_2}^{(1)} = 0$$

by the minimality of u. Thus there is a K_2-isomorphism of W_1 which maps $w_{i_1}^{(1)}$ to $w_{i_2}^{(1)}$ and therefore $w_{i_1}^{(1)}$ and $w_{i_2}^{(1)}$ are linearly dependent, as $\mathrm{End}_{\mathrm{K}_2}(\mathrm{W}_1) = \mathbf{C}$, and the minimality of u implies $i_1 = i_2$. Thus $u = 1$, $\mathrm{W}_1 \otimes w_1^{(2)}$ is contained in $\tilde{\mathrm{W}}$ (because W_1 is irreducible over K_2), and thus $\mathrm{W}_1 \otimes \mathrm{W}_2 \subseteq \tilde{\mathrm{W}}$. □

Proof of Proposition 6.4. The existence assertion (i) is proven mutatis mutandis to [42, Theorem 4.1]. Assertion (ii) follows from Corollary 4.9. For (iii): Let Λ_m be a self-dual $\mathfrak{o}_{\mathrm{E}}$-$\mathfrak{o}_{\mathrm{D}}$-lattice sequence such that $\tilde{\mathfrak{b}}(\Lambda_m)$ is minimal and $\tilde{\mathfrak{a}}(\Lambda_m) \subseteq \tilde{\mathfrak{a}}(\Lambda_{\mathrm{M}})$. The representation $\eta_{\Lambda_m,\Lambda_{\mathrm{M}}}$ is intertwined by the whole of G_β, by Proposition 4.5, and further the pro-p-Sylow subgroups of $\mathrm{J}_{\Lambda_{\mathrm{M}}}$ are all conjugate in $\mathrm{P}(\Lambda_{\mathrm{M},\beta})$. Thus (iii)(a) implies (iii)(b). Suppose (iii)(b) then, by Lemma 6.5(ii) $\kappa|_{\mathrm{J}^1_{\Lambda_m,\Lambda_{\mathrm{M}}}}$ is equivalent to $\eta_{\Lambda_m,\Lambda_{\mathrm{M}}} \otimes \varphi$ for some inflation φ of a characters of $\mathrm{J}^1_{\Lambda_m,\Lambda_{\mathrm{M}}}/\mathrm{J}^1_{\Lambda_{\mathrm{M}}}$ (The latter group is isomorphic to $\mathrm{P}_1(\Lambda_{m,\beta})/\mathrm{P}_1(\Lambda_{\mathrm{M},\beta})$). Thus φ is intertwined by the whole of G_β and by the analogue of [42, 3.10] we obtain that φ is trivial. □

Corollary 6.6. The sets $\mathrm{ext}(\Lambda, \Lambda')$ and β-$\mathrm{ext}(\Lambda_{\mathrm{M}})$ are not empty.

6.3. **The map $\Psi_{\Lambda,\Lambda',\Lambda''}$ in the inclusion case.** At first we assume $\tilde{\mathfrak{a}}(\Lambda) \subseteq \tilde{\mathfrak{a}}(\Lambda') \cap \tilde{\mathfrak{a}}(\Lambda'') \in \{\tilde{\mathfrak{a}}(\Lambda'), \tilde{\mathfrak{a}}(\Lambda'')\}$. Take $\kappa'_{\cong} \in \mathrm{ext}(\Lambda, \Lambda')$.

Lemma 6.7 (cf. [42] 4.3)**.** There a unique $(\kappa''_{\cong}, \mathrm{J}_{\Lambda,\Lambda''}) \in \mathrm{ext}(\Lambda, \Lambda'')$ such that

$$\mathrm{ind}_{\mathrm{J}_{\Lambda,\Lambda'}}^{\mathrm{P}_{\Lambda,\Lambda}} \kappa' \cong \mathrm{ind}_{\mathrm{J}_{\Lambda,\Lambda''}}^{\mathrm{P}_{\Lambda,\Lambda}} \kappa''. \tag{6.8}$$

(Where we define $\mathrm{P}_{\Lambda,\Lambda'} := \mathrm{P}(\Lambda_\beta)\mathrm{P}_1(\Lambda')$, e.g. $\mathrm{P}_{\Lambda,\Lambda} := \mathrm{P}(\Lambda_\beta)\mathrm{P}_1(\Lambda)$.)

Proof. The proof follows the idea of [21, Lemma 6.2] ([11, 5.2.5] and [42, 4.3]). By transitivity we only need to proof the assertion for the cases $(\Lambda, \Lambda', \Lambda'') = (\Lambda, \Lambda', \Lambda)$ and $(\Lambda, \Lambda', \Lambda'') = (\Lambda, \Lambda, \Lambda'')$. We just consider the first case. We have to find $\kappa_{\cong} \in \mathrm{ext}(\Lambda, \Lambda)$ such that (6.8) holds (for $\kappa'' = \kappa$), and it will be unique, because η_Λ has multiplicity one in the left side π of (6.8). Here is the outline: The restriction $\pi|_{\mathrm{P}_1(\Lambda)}$, which is $\mathrm{ind}_{\mathrm{J}^1_{\Lambda,\Lambda'}}^{\mathrm{P}_1(\Lambda)} \eta_{\Lambda,\Lambda'}$ is irreducible, because $\mathrm{I}_{\mathrm{P}_1(\Lambda)}(\eta_{\Lambda,\Lambda'})$ is $\mathrm{J}^1_{\Lambda,\Lambda'}$ (the group $\mathrm{P}_1(\Lambda)$ is also pro-p). Consider $\mathrm{Hom}_{\mathrm{J}^1_\Lambda}(\eta_\Lambda, \pi)$. From Proposition 4.5 and Mackey decomposition follows that η_Λ occurs in π with multiplicity one. We choose $\kappa = \pi^{\eta_\Lambda}$, see Lemma 5.1. We obtain (6.8) for $\kappa'' = \kappa$ because $\mathrm{ind}_{\mathrm{J}_\Lambda}^{\mathrm{P}_{\Lambda,\Lambda}} \kappa$ is irreducible by an argument similar to the one given at the beginning of the proof. □

We define $\Psi_{\Lambda,\Lambda',\Lambda''}(\kappa'_{\cong}) := \kappa''_{\cong}$ using $\kappa''_{\cong}$ from Lemma 6.7. In fact $\mathrm{J}_{\Lambda,\Lambda'}$ does only depend on $\tilde{\mathfrak{b}}(\Lambda)$ instead of Λ, and even more:

Lemma 6.9. Suppose $\tilde{\Lambda} \in \mathrm{Latt}^1_{\mathfrak{o}_\mathrm{E},\mathfrak{o}_\mathrm{D}}(\mathrm{V})$. Then $\Psi_{\Lambda,\Lambda',\Lambda''} \circ \mathrm{Res}^{\mathrm{J}_{\tilde{\Lambda},\Lambda'}}_{\mathrm{J}_{\Lambda,\Lambda'}}$ and $\mathrm{Res}^{\mathrm{J}_{\tilde{\Lambda},\Lambda''}}_{\mathrm{J}_{\Lambda,\Lambda''}} \circ \Psi_{\tilde{\Lambda},\Lambda',\Lambda''}$ coincide if $\tilde{\mathfrak{b}}(\Lambda) \subseteq \tilde{\mathfrak{b}}(\tilde{\Lambda})$ and $\tilde{\mathfrak{a}}(\tilde{\Lambda}) \subseteq \tilde{\mathfrak{a}}(\Lambda') \cap \tilde{\mathfrak{a}}(\Lambda'')$.

Proof. To show this assertion it is enough to consider the case $\tilde{\mathfrak{a}}(\Lambda) \subseteq \tilde{\mathfrak{a}}(\tilde{\Lambda})$. (For the general case take $\bar{\Lambda} \in \mathrm{Latt}_{\mathfrak{o}_\mathrm{E},\mathfrak{o}_\mathrm{D}}\mathrm{V}$ with $\tilde{\mathfrak{a}}(\bar{\Lambda}) \subseteq \tilde{\mathfrak{a}}(\tilde{\Lambda})$ and $\tilde{\mathfrak{b}}(\Lambda) = \tilde{\mathfrak{b}}(\bar{\Lambda})$, and use a segment from Λ to $\bar{\Lambda}$.) We start with (κ', κ'') satisfying (6.8) for $\tilde{\Lambda}$ instead of Λ. Then we restrict to $\mathrm{P}_{\Lambda,\tilde{\Lambda}}$ and induce to $\mathrm{P}_{\Lambda,\Lambda}$ to obtain (6.8) for Λ. This proves the lemma. □

By Lemma 6.9 we can define $\Psi_{\Lambda,\Lambda',\Lambda''}$ if $\tilde{\mathfrak{a}}(\Lambda)$ may not be contained in $\tilde{\mathfrak{a}}(\Lambda') \cap \tilde{\mathfrak{a}}(\Lambda'')$. Suppose $\tilde{\mathfrak{b}}(\Lambda) \subseteq \tilde{\mathfrak{a}}(\Lambda') \cap \tilde{\mathfrak{a}}(\Lambda'')$ and choose $\tilde{\Lambda}$ such that $\tilde{\mathfrak{a}}(\tilde{\Lambda}) \subseteq \tilde{\mathfrak{a}}(\Lambda') \cap \tilde{\mathfrak{a}}(\Lambda'')$ and $\tilde{\mathfrak{b}}(\Lambda) = \tilde{\mathfrak{b}}(\tilde{\Lambda})$. Then define $\Psi_{\Lambda,\Lambda',\Lambda''}$ to be $\Psi_{\tilde{\Lambda},\Lambda',\Lambda''}$.

6.4. **The map $\Psi_{\Lambda,\Lambda',\Lambda''}$ in the general case.** We do not require $\tilde{\mathfrak{a}}(\Lambda') \cap \tilde{\mathfrak{a}}(\Lambda'') \in \{\tilde{\mathfrak{a}}(\Lambda'), \tilde{\mathfrak{a}}(\Lambda'')\}$ here. We choose a path (6.1) of self-dual $\mathfrak{o}_\mathrm{E}$-$\mathfrak{o}_\mathrm{D}$-lattice sequences and define $\Psi_{\Lambda,\Lambda',\Lambda''}$ as in (6.3). Now one has to prove that this definition is independend of the choice of the path. For that it is enough to consider a triangle of self-dual $\mathfrak{o}_\mathrm{E}$-$\mathfrak{o}_\mathrm{D}$ lattice sequences $\Lambda_1, \Lambda_2, \Lambda_3$ such that $\tilde{\mathfrak{a}}(\Lambda_1) \subseteq \tilde{\mathfrak{a}}(\Lambda_2) \subseteq \tilde{\mathfrak{a}}(\Lambda_3)$ with $\tilde{\mathfrak{b}}(\Lambda) \subseteq \tilde{\mathfrak{b}}(\Lambda_1)$ and show the commutativity

$$\Psi_{\Lambda,\Lambda_2,\Lambda_3} \circ \Psi_{\Lambda,\Lambda_1,\Lambda_2} = \Psi_{\Lambda,\Lambda_1,\Lambda_3}.$$

We take an $\mathfrak{o}_\mathrm{E}$-$\mathfrak{o}_\mathrm{D}$-lattice sequence $\tilde{\Lambda}$ such that $\tilde{\mathfrak{a}}(\tilde{\Lambda}) \subseteq \tilde{\mathfrak{a}}(\Lambda_1)$ and $\tilde{\mathfrak{b}}(\Lambda) = \tilde{\mathfrak{b}}(\tilde{\Lambda})$. We choose $\kappa_{i,\cong} \in \mathrm{ext}(\Lambda, \Lambda_i)$, $i = 1, 2, 3$, such that $\Psi_{\Lambda,\Lambda_1,\Lambda_2}(\kappa_{1,\cong}) = \kappa_{2,\cong}$ and $\Psi_{\Lambda,\Lambda_1,\Lambda_3}(\kappa_{1,\cong}) =$

$\kappa_{3,\cong}$. Then $\Psi_{\Lambda,\Lambda_2,\Lambda_3}(\kappa_{2,\cong}) = \kappa_{3,\cong}$ follows from (6.8) and transitivity. This finishes the definition of $\Psi_{\Lambda,\Lambda',\Lambda''}$. We have the following result on intertwining:

Proposition 6.10 (cf. [42] Lemma 4.3)**.** Suppose $\Psi_{\Lambda,\Lambda',\Lambda''}(\kappa'_{\cong}) = \kappa''_{\cong}$. Then $\mathrm{I}_{\mathrm{G}_\beta}(\kappa') = \mathrm{I}_{\mathrm{G}_\beta}(\kappa'')$.

For this proposition we need the following intersection property:

Lemma 6.11 (cf.[42] 2.6)**.** Let g be an element of G_β. Then we have

$$(\mathrm{P}_1(\Lambda)g\mathrm{P}_1(\Lambda)) \cap \mathrm{G}_\beta = \mathrm{P}_1(\Lambda_\beta)g\mathrm{P}_1(\Lambda_\beta). \tag{6.12}$$

Proof. At first: The proof of

$$(\tilde{\mathrm{P}}^1(\Lambda)g\tilde{\mathrm{P}}^1(\Lambda)) \cap \tilde{\mathrm{G}}_\beta = \tilde{\mathrm{P}}^1(\Lambda_\beta)g\tilde{\mathrm{P}}^1(\Lambda_\beta) \tag{6.13}$$

is mutatis mutanadis to the proof of **II**.4.8. Now one takes σ-fixed points on both sides of (6.13) to obtain (6.12) by [21, 2.12(i)]. □

Proof of Proposition 6.10. Using the construction of $\Psi_{\Lambda,\Lambda',\Lambda''}$ we can assume without loss of generality that $\tilde{\mathfrak{a}}(\Lambda) \subseteq \tilde{\mathfrak{a}}(\Lambda') \subseteq \tilde{\mathfrak{a}}(\Lambda'')$. Now the proof is as for the second part of [42, 4.3] (see [30, 2.9]), using Lemma 6.11 instead of [42, 2.6]. □

6.5. **Defining β-extensions in the general case.** Suppose further that $\tilde{\mathfrak{b}}(\Lambda'')$ is contained in $\tilde{\mathfrak{b}}(\Lambda_{\mathrm{M}})$ in this paragraph.

Definition 6.14 (cf.[42] 4.5, 4.7)**.** Granted $\tilde{\mathfrak{b}}(\Lambda) \subseteq \tilde{\mathfrak{b}}(\Lambda_{\mathrm{M}})$, we call the following set the *set of (equivalence classes) of β-extensions of η_Λ to J_Λ relative to Λ_{M}*:

$$\beta\text{-ext}_{\Lambda_{\mathrm{M}}}(\Lambda) := \{\Psi_{\Lambda,\Lambda_{\mathrm{M}},\Lambda}(\mathrm{Res}^{\mathrm{J}_{\Lambda_{\mathrm{M}}}}_{\mathrm{J}_{\Lambda,\Lambda_{\mathrm{M}}}} \kappa_{\cong})|\ \kappa_{\cong} \in \beta\text{-ext}(\Lambda_{\mathrm{M}})\}.$$

We call the elements of

$$\beta\text{-ext}^0_{\Lambda_{\mathrm{M}}}(\Lambda) := \mathrm{Res}^{\mathrm{J}_\Lambda}_{\mathrm{J}^0_\Lambda}(\beta\text{-ext}_{\Lambda_{\mathrm{M}}}(\Lambda))$$

(equivalence classes) of *β-extensions of η_Λ to J^0_Λ relative to* Λ_{M}.

Theorem 6.15 ([42] 4.10)**.** Granted $\tilde{\mathfrak{b}}(\Lambda') \cup \tilde{\mathfrak{b}}(\Lambda'') \subseteq \tilde{\mathfrak{b}}(\Lambda_{\mathrm{M}})$, there is a unique map $\Psi^0_{\Lambda',\Lambda''}$ from $\beta\text{-ext}^0_{\Lambda_{\mathrm{M}}}(\Lambda')$ to $\beta\text{-ext}^0_{\Lambda_{\mathrm{M}}}(\Lambda'')$ depending on Λ_{M} such that

$$\Psi^0_{\Lambda',\Lambda''} \circ \mathrm{Res}^{\mathrm{J}_{\Lambda'}}_{\mathrm{J}^0_{\Lambda'}} \circ \Psi_{\Lambda',\Lambda_{\mathrm{M}},\Lambda'} \circ \mathrm{Res}^{\mathrm{J}_{\Lambda_{\mathrm{M}}}}_{\mathrm{J}_{\Lambda',\Lambda_{\mathrm{M}}}} = \mathrm{Res}^{\mathrm{J}_{\Lambda''}}_{\mathrm{J}^0_{\Lambda''}} \circ \Psi_{\Lambda'',\Lambda_{\mathrm{M}},\Lambda''} \circ \mathrm{Res}^{\mathrm{J}_{\Lambda_{\mathrm{M}}}}_{\mathrm{J}_{\Lambda'',\Lambda_{\mathrm{M}}}} \tag{6.16}$$

on $\beta\text{-ext}(\Lambda_{\mathrm{M}})$. The map $\Psi^0_{\Lambda',\Lambda''}$ is bijective.

Proof. At first: A map $\Psi^0_{\Lambda',\Lambda''}$ satisfying (6.16) is uniquely determined and surjective by the definition of $\beta\text{-ext}^0_{\Lambda_{\mathrm{M}}}(\Lambda')$ and $\beta\text{-ext}^0_{\Lambda_{\mathrm{M}}}(\Lambda'')$. Further we have $\Psi^0_{\Lambda',\Lambda'} = \mathrm{id}_{\beta\text{-ext}^0_{\Lambda_{\mathrm{M}}}(\Lambda')}$ and if $\Psi^0_{\Lambda',\Lambda''}$ and $\Psi^0_{\Lambda'',\Lambda'''}$ satisfy (6.16) then $\Psi^0_{\Lambda'',\Lambda'''} \circ \Psi^0_{\Lambda',\Lambda''}$ too. Further if $\Psi^0_{\Lambda',\Lambda''}$ exists and is bijective then we can take $(\Psi^0_{\Lambda',\Lambda''})^{-1}$ as $\Psi^0_{\Lambda'',\Lambda'}$. Thus we only have to consider the case $\tilde{\mathfrak{a}}(\Lambda') \subseteq \tilde{\mathfrak{a}}(\Lambda'')$. We define $\Psi^0_{\Lambda',\Lambda''}$ in several steps:

- Let κ'^0 be a β-extension of $\eta_{\Lambda'}$ to $\mathrm{J}^0_{\Lambda'}$ and let κ' be an arbitrary β-extension of $\eta_{\Lambda'}$ to $\mathrm{J}_{\Lambda'}$ such that the restriction of κ' to $\mathrm{J}^0_{\Lambda'}$ is equivalent to κ'^0.
- We choose a class $\kappa_{\mathrm{M},\cong} \in \beta\text{-ext}(\Lambda_{\mathrm{M}})$ such that

$$\Psi_{\Lambda',\Lambda_{\mathrm{M}},\Lambda'}(\mathrm{Res}^{\mathrm{J}_{\Lambda_{\mathrm{M}}}}_{\mathrm{J}_{\Lambda',\Lambda_{\mathrm{M}}}} \kappa_{\mathrm{M},\cong}) = \kappa'_{\cong}$$

  and we put $\kappa''_{\cong} := \Psi_{\Lambda'',\Lambda_{\mathrm{M}},\Lambda''}(\mathrm{Res}^{\mathrm{J}_{\Lambda_{\mathrm{M}}}}_{\mathrm{J}_{\Lambda'',\Lambda_{\mathrm{M}}}} \kappa_{\mathrm{M},\cong})$.
- Then we define:

$$\Psi^0_{\Lambda',\Lambda''}(\kappa'^0_{\cong}) := \kappa''|_{\mathrm{J}^0_{\Lambda''}} =: \kappa''^0_{\cong}.$$

We claim that $\Psi^0_{\Lambda',\Lambda''}$ is well-defined, i.e. independent of the choices made. Denote

$$\kappa_{\cong} := \Psi_{\Lambda',\Lambda',\Lambda''}(\kappa'_{\cong}).$$

Then we obtain by definition and Lemma 6.9 (see the paragraph after 6.9 to get the analogue for the general Ψ, see §6.4)

$$\begin{aligned}
\kappa_{\cong} &= (\Psi_{\Lambda',\Lambda_{\mathrm{M}},\Lambda''} \circ \Psi_{\Lambda',\Lambda',\Lambda_{\mathrm{M}}})(\kappa'_{\cong}) \\
&= \Psi_{\Lambda',\Lambda_{\mathrm{M}},\Lambda''}(\mathrm{Res}^{\mathrm{J}_{\Lambda_{\mathrm{M}}}}_{\mathrm{J}_{\Lambda',\Lambda_{\mathrm{M}}}} \kappa_{\mathrm{M},\cong}) \\
&= (\mathrm{Res}^{\mathrm{J}_{\Lambda''}}_{\mathrm{J}_{\Lambda',\Lambda''}} \circ \Psi_{\Lambda'',\Lambda_{\mathrm{M}},\Lambda''})(\mathrm{Res}^{\mathrm{J}_{\Lambda_{\mathrm{M}}}}_{\mathrm{J}_{\Lambda'',\Lambda_{\mathrm{M}}}} \kappa_{\mathrm{M},\cong}) \\
&= \mathrm{Res}^{\mathrm{J}_{\Lambda''}}_{\mathrm{J}_{\Lambda',\Lambda''}} \kappa''_{\cong}.
\end{aligned}$$

Thus (6.8) (with $\Lambda = \Lambda'$) is satisfied. We restrict (6.8) to $\mathrm{P}^0_{\Lambda',\Lambda'}$ (This is $\mathrm{P}^0(\Lambda'_\beta)\mathrm{P}_1(\Lambda')$) to obtain:

$$\mathrm{ind}^{\mathrm{P}^0_{\Lambda',\Lambda'}}_{\mathrm{J}^0_{\Lambda',\Lambda''}} \kappa''|_{\mathrm{J}^0_{\Lambda',\Lambda''}} \cong \mathrm{ind}^{\mathrm{P}^0_{\Lambda',\Lambda'}}_{\mathrm{J}^0_{\Lambda'}} \kappa'^0 \tag{6.17}$$

Both sides are irreducible, because their restrictions to $\mathrm{P}^1_{\Lambda'}$ are. This implies that κ'^0 uniquely determines the isomorphism class of the restriction of $\kappa''^0_{\cong}$ to $\mathrm{J}^0_{\Lambda',\Lambda''}$, because, as in the proof of Lemma 6.7, $\eta_{\Lambda',\Lambda''}$ has multiplicity one in the left hand side of (6.17). Now mutatis mutandis as in the proof of [42, 4.10] one shows that there is only one element of $\beta\text{-ext}^0_{\Lambda_{\mathrm{M}}}(\Lambda'')$ extending $\mathrm{Res}^{\mathrm{J}^0_{\Lambda''}}_{\mathrm{J}^0_{\Lambda',\Lambda''}} \kappa''^0_{\cong}$. Thus $\kappa''^0_{\cong}$ is uniquely determined by $\kappa'^0_{\cong}$. This shows that $\Psi^0_{\Lambda',\Lambda''}$ is well-defined. On the other hand the restriction of κ'' to $\mathrm{J}^0_{\Lambda',\Lambda''}$ uniquely determines $\kappa'^0_{\cong}$, by equation (6.17). Hence the injectivity of $\Psi^0_{\Lambda',\Lambda''}$. □

7. Cuspidal types

In this section we construct cuspidal types for G, similar to [42] and [21] for $\mathrm{G} \otimes \mathrm{L}$ (cf. [11], [30]), and we follow their proofs. Let $\theta \in \mathrm{C}(\Delta)$ with $r = 0$ be a self-dual semisimple character with Heisenberg representation (η, J^1). From now on we skip the lattice sequence from the subscript if there is no cause of confusion, e.g. we write $\mathrm{J}^1, \mathrm{J}, \eta$ for $\mathrm{J}^1_\Lambda, \mathrm{J}_\Lambda, \eta_\Lambda$.

Definition 7.1 (cf. [24] Definition 3.3). Let ρ be an irreducible **C**-representations of $\mathrm{P}(\Lambda_\beta)$ whose restriction to $\mathrm{P}^0(\Lambda_\beta)$ is an inflation of a direct sum of cuspidal irreducible representations of $\mathbb{P}(\Lambda_\beta)^0(\mathrm{k_F})$. We call such a ρ a *cuspidal inflation* w.r.t. (Λ, β). Let κ

be a β-extension of η (with respect to some Λ_{M} with $\tilde{\mathfrak{b}}(\Lambda_{\mathrm{M}})$ maximal). We call the representation $\lambda := \kappa \otimes \rho$ a *cuspidal type* of G if

- the parahoric $\mathrm{P}^0(\Lambda_\beta)$ is maximal and
- the centre of G_β is compact.

Remark 7.2. If λ is a cuspidal type then the underlying stratum Δ has to be skew, i.e. the action of σ_h on the index set I is trivial, because of the compactness of the centre of G_β.

The main motivation for the definition of a β-extension is the following theorem, which is assertion (ii) in Theorem 1.1:

Theorem 7.3 (cf. [42, Theorem 6.18], [21, Theorem 12.1])**.** Let λ be a cuspidal type. Then $\mathrm{ind}_{\mathrm{J}}^{\mathrm{G}} \lambda$ is a cuspidal irreducible representation of G.

The proof in case that **C** has positive characteristic needs an extra lemma. We write $\mathrm{mult}_\eta(\pi)$ for the multiplicity of η in a smooth representation π.

Lemma 7.4 (cf. [21] Corollary 8.5(ii))**.** Let $(\lambda = \kappa \otimes \rho, \mathrm{J})$ be a cuspidal type. Put $\pi = \mathrm{ind}_{\mathrm{J}}^{\mathrm{G}} \rho$. Then

(i) $\mathrm{Hom}_{\mathrm{J}^1}(\kappa, \pi) \simeq \rho$ over J and
(ii) $\pi^\eta = \lambda$ (λ is canonically contained in π).

Proof. (i) The proof of [21, Corollary 8.5(ii)] is valid without changes in our situation.
(ii) The representation (λ, J) is contained in π^η and

$$\mathrm{mult}_\eta(\pi^\eta) = \dim_{\mathbf{C}} \mathrm{Hom}_{\mathrm{J}^1}(\eta, \pi) = \dim_{\mathbf{C}}(\rho) = \mathrm{mult}_\eta(\lambda)$$

by (i). This finishes the proof.

□

Proof of Theorem 7.3. Let $\lambda = \kappa \otimes \rho$ be a cuspidal type. Put $\pi := \mathrm{ind}_{\mathrm{J}}^{\mathrm{G}} \lambda$. We want to apply the irreducibility criterion [45, 4.2] to show that π is irreducible and hence cuspidal irreducible. We have to show two parts:

Part 1: $\mathrm{I}_{\mathrm{G}}(\lambda) = \mathrm{J}$ (which already implies the irreducibility of π if **C** has characteristic zero) and
Part 2: The representation λ is a direct summand of every smooth irreducible representation of G whose restriction to J has λ as a subrepresentation.

Part 1: The proof of this part is similar to the proof of [42, Proposition 6.18], but we are going to give a simplification avoiding the use of [42, Corollary 6.16].

Take an irreducible component (ρ_0, W_{ρ_0}) of $\rho|_{\mathrm{J}^0}$ and denote $\kappa_0 = \kappa|_{\mathrm{J}^0}$. Then $\kappa_0 \otimes \rho_0$ is irreducible by Lemma 6.5(i). The restriction of λ to J^0 is equivalent to a direct sum of J/J^0-conjugates of $\kappa_0 \otimes \rho_0$, because J^0 is a normal subgroup of J. Note that J/J^0 is isomorphic to $\mathrm{P}(\Lambda_\beta)/\mathrm{P}^0(\Lambda_\beta)$, so that the conjugating elements can be taken in $\mathrm{P}(\Lambda_\beta)$. An element $g \in \mathrm{G}$ which intertwines λ intertwines $\kappa_0 \otimes \rho_0$ up to $\mathrm{P}(\Lambda_\beta)$-conjugation, and it also intertwines η. So it is an element of $\mathrm{J}^1\mathrm{G}_\beta\mathrm{J}^1$. We can therefore without loss of generality assume g as an element of G_β which intertwines $\kappa_0 \otimes \rho_0$. Hence as $\mathrm{I}_g(\eta)$ is one-dimensional and the restriction of ρ_0 to J^1 is trivial we obtain that a g-intertwiner of $\kappa_0 \otimes \rho_0$, i.e. an non-zero element of $\mathrm{I}_g(\kappa_0 \otimes \rho_0)$, has to be a tensor product of endomorphisms $S \in \mathrm{I}_g(\eta)$ and $T \in \mathrm{End}_{\mathbf{C}}(\mathrm{W}_{\rho_0})$, see [21, Lemma 2.7]. Let Q be a pro-p-Sylow subgroup of J^0. Then g is an element of $\mathrm{I}(\kappa|_Q)$ by the definition of β-extension. In particular $S \in \mathrm{I}_g(\kappa|_Q)$, because $\mathrm{I}_g(\eta)$ is 1-dimensional. Thus $T \in \mathrm{I}_g(\rho_0|_Q)$. In particular g intertwines the restriction of ρ to a pro-p-Sylow subgroup. Thus, by Morris theory, i.e. here [21, Lemma 7.4] and [42, Proposition 1.1(ii)], g is an element of $\mathrm{P}(\Lambda_\beta)$. This finishes the proof of part one.

Part 2: The proof is given in the proof of [21, Theorem 12.1]: An irreducible representation π' containing λ is a quotient of π by Frobenius reciprocity and we therefore have

$$\lambda = \pi^\eta \twoheadrightarrow \pi'^\eta \supseteq \lambda$$

by Lemma 7.4(ii). Thus $\pi'^\eta = \lambda$ and Lemma 5.1 finishes the proof. □

8. Partitions subordinate to a stratum

In the proof of the exhaustion in [42] the author has to pass to decompositions of V which are so-called exactly subordinate to a skew-semisimple stratum Δ (with $r = 0$), see [42, Definition 6.5]. In our situation of quaternionic forms we need to generalize this approach, because the centralizer of E_i in $\mathrm{End}_\mathrm{D}\,\mathrm{V}^i$ is not given by the same vector space V^i, if $\beta_i \neq 0$, i.e. $\mathrm{End}_{\mathrm{E}_i\otimes\mathrm{D}}\,\mathrm{V}^i$ is isomorphic $\mathrm{End}_{\mathrm{D}^i_\beta}\,\mathrm{V}^i_\beta$, see §2.4 (We have $2\dim_{\mathrm{E}_i}(\mathrm{V}^i_\beta) = \dim_{\mathrm{E}_i} V^i$). We generalize the notion of decompositions of V which are exactly subordinate to a semisimple stratum by certain families of idempotents. (This is indicated by the arguments given in [42, §5].) We fix a semisimple stratum Δ with $r = 0$.

Definition 8.1. (i) We call a finite tuple of idempotents $(e^{(j)})_j$ of $\mathrm{B} = \mathrm{End}_{\mathrm{E}\otimes_\mathrm{F}\mathrm{D}}(\mathrm{V})$ an $\mathrm{E}\otimes_\mathrm{F}\mathrm{D}$-*partition* of V if $e^{(j)}e^{(k)} = \delta_{jk}$ for all $j \neq k$ and $\sum_j e^{(j)} = 1$. An $\mathrm{E}\otimes_\mathrm{F}\mathrm{D}$-partition $(e^{(j)})_j$ is called a *subordinate* to Δ if $(\mathrm{W}^{(j)})_j$ with $\mathrm{W}^{(j)} := e^{(j)}\mathrm{V}$ is a splitting of Δ, or equivalently if $(\mathrm{W}^{(j)})_j$ is a splitting of Λ, i.e. $e^{(j)} \in \tilde{\mathfrak{a}}(\Lambda)$ for all j.

(ii) We call an $\mathrm{E}\otimes_\mathrm{F}\mathrm{D}$-partition $(e^{(j)})_j$ of V *properly subordinate* to Δ if it is subordinate to Δ and the residue class $e^{(j)} + \tilde{\mathfrak{b}}_1(\Lambda)$ in $\tilde{\mathfrak{b}}(\Lambda)/\tilde{\mathfrak{b}}_1(\Lambda)$ is a central idempotent.

Analogously we have the notion of "being self-dual-subordinate to a stratum":

Definition 8.2. Suppose Δ is a skew semisimple stratum. Let $(e^{(j)})_j$ be an $\mathrm{E}\otimes_\mathrm{F}\mathrm{D}$-partition of V subordinate to Δ. The partition $(e^{(j)})_j$ is called *self-dual-subordinate* to Δ if the set of the idempotents $e^{(j)}$ is σ_h-invariant with at most one fixed point. As in [42] we are then going to use $\{0, \pm 1, \ldots, \pm m\}$ as the index set, such that $\sigma_h(e^{(j)}) = e^{(-j)}$ for all j. (We just have added $e^{(0)} := 0$ if there was no σ_h-fixed idempotent among the $e^{(j)}$.)

An $E\otimes_F D$-partition self-dual-subordinate to Δ is called *properly self-dual-subordinate* to Δ, if the partition is properly subordinate to Δ. Suppose $(e^{(j)})_j$ is properly self-dual-subordinate to Δ. We call it *exactly subordinate to* Δ if it cannot be refined by another $E\otimes_F D$-partition of V properly self-dual-subordinate to Δ.

These notions of partitions subordinate to a stratum enable Iwahori decompositions as in [42]. Let $(e^{(j)})_j$ be a $E\otimes_F D$-partition of V. Let $\tilde{M}$ be the Levi subgroup of $\tilde{G}$ defined as:

$$\tilde{M} := \tilde{G}\cap(\prod_j \mathrm{End}_D(W^{(j)})).$$

Let $\tilde{P}$ be a parabolic subgroup of $\tilde{G}$ with Levi $\tilde{M}$, and write $\tilde{U}_+$ and $\tilde{U}_-$ for the radical of $\tilde{P}$ and the opposite parabolic $\tilde{P}^{op}$, respectively. We write M, P, U_+ and U_- for the corresponding intersections with G.

Lemma 8.3 (cf. [42] 5.2, 5.10)**.** Suppose $(e^{(j)})_j$ is subordinate to Δ.

(i) Then $\tilde{H}^1(\beta,\Lambda)$ and $\tilde{J}^1(\beta,\Lambda)$ have Iwahori decompositions with respect to the product $\tilde{U}_-\tilde{M}\tilde{U}_+$. Moreover, the groups $\tilde{H}(\beta,\Lambda)$ and $\tilde{J}(\beta,\Lambda)$ have a Iwahori decomposition with respect to $\tilde{U}_-\tilde{M}\tilde{U}_+$ if $(e^{(j)})_j$ is properly subordinate to Δ.

(ii) Suppose Δ is skew-semisimple and that $(e^{(j)})_j$ is self-dual-subordinate to Δ. Then the groups $H^1(\beta,\Lambda)$ and $J^1(\beta,\Lambda)$ have Iwahori decompositions with respect to U_-MU_+. Additionally, $H(\beta,\Lambda)$ and $J(\beta,\Lambda)$ have a Iwahori decomposition with respect to U_-MU_+ if $(e^{(j)})_j$ is properly self-dual-subordinate to Δ.

Proof. We just show the first assertion of (i), because the other statements follow similarly. The idempotents satisfy $e^{(j)}\in\tilde{\mathfrak{b}}(\Lambda)\subseteq\tilde{\mathfrak{b}}(\Lambda_L)$. We apply *loc.cit.* to obtain for $\tilde{M}_L = \tilde{M}\otimes L, \tilde{U}_{+,L}=\tilde{U}_+\otimes L$ and $\tilde{U}_{-,L}=\tilde{U}_-\otimes L$:

$$\tilde{H}^1(\beta,\Lambda)\subseteq(\tilde{H}^1(\beta\otimes 1,\Lambda_L)\cap\tilde{U}_{-,L})(\tilde{H}^1(\beta\otimes 1,\Lambda_L)\cap\tilde{M}_L)(\tilde{H}^1(\beta\otimes 1,\Lambda_L)\cap\tilde{U}_{+,L}).$$

The τ-invariance of the three factors and the uniqueness of the Iwahori decomposition (w.r.t. $\tilde{U}_{-,L}\tilde{M}_L\tilde{U}_{+,L}$) gives the result. □

Suppose that Δ is skew-semisimple and that $(e^{(j)})_j$ is properly self-dual-subordinate to Δ. Let (η, J^1) be the Heisenberg representation of a self-dual semisimple character θ and let κ be a β-extension of η. As in §11, as Δ is fixed, we skip the parameters Λ and β for the sets H^1, J^1, J, etc.. Analogously to [42] we can introduce representations (θ_P), (η_P, J^1_P) and (κ_P, J_P). The corresponding groups are defined via:

$$J^1_P = (H^1\cap U_-)(J^1\cap P) = (H^1\cap U_-)(J^1\cap M)(J^1\cap U_+)$$

and

$$J_P = (H^1\cap U_-)(J\cap P) = (H^1\cap U_-)(J\cap M)(J^1\cap U_+).$$

At first one extends θ to a character θ_P trivially to $H^1_P := (H^1\cap U_-)(H^1\cap M)(J^1\cap U_+)$, i.e. via

$$\theta_P(xy) := \theta(x),\ x\in H^1,\ y\in(J^1\cap U_+).$$

We define (η_P, J^1_P) as the natural representation (given by η) on the set of $(J^1\cap U_+)$-fixed point of η. Similarly, we define (κ_P, J_P) on the set of $J\cap U_+$-fixed points, using κ.

Then we have the following properties:

Proposition 8.4 (cf. [42] Lemma 5.12, Proposition 5.13 for $\mathrm{G}\otimes\mathrm{L}$)**.** κ_{P} is an extension of η_{P} and η_{P} is the Heisenberg representation of θ_{P} on $\mathrm{J}^1_{\mathrm{P}}$. Further we have $\mathrm{ind}_{\mathrm{J}^1_{\mathrm{P}}}^{\mathrm{J}^1}\eta_{\mathrm{P}}\cong\eta$ and $\mathrm{ind}_{\mathrm{J}_{\mathrm{P}}}^{\mathrm{J}}\kappa_{\mathrm{P}}\cong\kappa$.

By [41, Lemma 5.12] the natural representation $(\eta_{\mathrm{P}_{\mathrm{L}}},\mathrm{J}^1_{\mathrm{P}_{\mathrm{L}}})$ of η_{L} on the set of $\mathrm{J}^1_{\mathrm{P}_{\mathrm{L}}}\cap\mathrm{U}_{\mathrm{L},+}$-fixed points of η_{L} induces to η_{L} (for the complex case by *loc.cit.* and for $\mathbf{C}$ by the Brauer map, as $\mathrm{Br}(\eta^{\mathbf{C}}_{\mathrm{P}_{\mathrm{L}}})=\eta^{\mathbb{C}}_{\mathrm{P}_{\mathrm{L}}}$). We will use this in the Proof.

Proof. Note at first that $\mathrm{J}^1/\mathrm{H}^1$ is abelian, so subgroups in between H^1 and J^1 are normal in J^1. On the group $(\mathrm{J}^1\cap\mathrm{M})\mathrm{H}^1$, which we denote by $\mathrm{J}^1_{\mathrm{M}}$, is a Heisenberg representation η_{M} of θ. Every irreducible representation of $\mathrm{J}^1_{\mathrm{P}}$ containing θ is of the form $\eta_{\mathrm{M}}\otimes\phi$ for some inflation ϕ of a character of $\mathrm{J}^1_{\mathrm{P}}/\mathrm{J}^1_{\mathrm{M}}$ (this is isomorphic to $\mathrm{J}^1\cap\mathrm{U}_+/\mathrm{H}^1\cap\mathrm{U}_+$):

$$\eta_{\mathrm{M}}\otimes\phi(xy):=\eta_{\mathrm{M}}(x)\phi(y),\ x\in\mathrm{J}^1_{\mathrm{M}},\ y\in\mathrm{J}^1\cap\mathrm{U}_+\,.$$

So, up to equivalence, the only irreducible representation of $\mathrm{J}^1_{\mathrm{P}}$ containing θ and $\mathbf{1}_{\mathrm{J}^1\cap\mathrm{U}_+}$ (the trivial representation of $\mathrm{J}^1\cap\mathrm{U}_+$) is $\eta_{\mathrm{M}}\otimes\mathbf{1}_{\mathrm{J}^1\cap\mathrm{U}_+}$. Thus the Glauberman transfer $(\mathbf{gl}^{\mathrm{L}|\mathrm{F}}_{\mathbf{C}}(\eta_{\mathrm{P}_{\mathrm{L}}}),\mathrm{J}^1_{\mathrm{P}})$ is equivalent to $\eta_{\mathrm{M}}\otimes\mathbf{1}_{\mathrm{J}^1\cap\mathrm{U}^+}$. Now, if $g\in\mathrm{J}^1$ intertwines the latter representation, so it normalizes $\mathbf{gl}^{\mathrm{L}|\mathrm{F}}_{\mathbf{C}}(\eta_{\mathrm{P}_{\mathrm{L}}})$ and therefore, by the injectivity of the Glauberman transfer, g normalizes $\eta_{\mathrm{P}_{\mathrm{L}}}$ as well. Thus $g\in\mathrm{J}^1_{\mathrm{P}_{\mathrm{L}}}\cap\mathrm{G}=\mathrm{J}^1_{\mathrm{P}}$, by Mackey, as $\eta_{\mathrm{P}_{\mathrm{L}}}$ induces irreducibly to $\mathrm{J}^1_{\mathrm{L}}$, and we obtain

$$\mathrm{ind}_{\mathrm{J}^1_{\mathrm{P}}}^{\mathrm{J}^1}(\eta_{\mathrm{M}}\otimes\mathbf{1}_{\mathrm{J}^1\cap\mathrm{U}_+})\cong\eta,$$

as the left hand side is irreducible and contains θ. The restriction of η to $\mathrm{J}^1_{\mathrm{P}}$ is a direct sum of extensions of η_{M}, and further, $\eta_{\mathrm{M}}\otimes\mathbf{1}_{\mathrm{J}^1\cap\mathrm{U}_+}$ has multiplicity one in η, by Frobenius reciprocity. Thus, we conclude that η_{P} is equivalent to $\eta_{\mathrm{M}}\otimes\mathbf{1}_{\mathrm{J}^1\cap\mathrm{U}_+}$, and is therefore the Heisenberg representation of θ_{P}. This finishes the assertions corresponding to η_{P}.

We now prove the assertion for κ_{P}. This representation is irreducible, because its restriction to $\mathrm{J}^1_{\mathrm{P}}$ is equivalent to η_{P}, as $\mathrm{J}^1\cap\mathrm{U}_+=\mathrm{J}\cap\mathrm{U}_+$ (The decomposition is properly subordinate to Δ!). Therefore, we have

$$\mathrm{Res}^{\mathrm{J}}_{\mathrm{J}^1}(\mathrm{ind}^{\mathrm{J}}_{\mathrm{J}_{\mathrm{P}}}\kappa_{\mathrm{P}})\cong\mathrm{ind}_{\mathrm{J}^1_{\mathrm{P}}}^{\mathrm{J}^1}\eta_{\mathrm{P}}\cong\eta,$$

so κ_{P} induces irreducibly to κ. □

The restrictions of θ, η_{P} and κ_{P} to M are tensor-products

$$\theta|_{\mathrm{H}^1\cap\mathrm{M}}=\otimes_j\theta_j,\ \eta_{\mathrm{P}}|_{\mathrm{J}^1\cap\mathrm{M}}=\otimes_j\eta_j,\ \kappa_{\mathrm{P}}|_{\mathrm{J}\cap\mathrm{M}}=\otimes_j\kappa_j,$$

where η_j is the Heisenberg representation of the semisimple characters θ_j.

9. Main theorems for the classification

Given a cuspidal irreducible representation of G then there is a semisimple character $\theta \in \mathrm{C}(\Lambda, 0, \beta)$ contained in π. Thus it contains the Heisenberg representation (η, J^1) of θ and there is an irreducible representation ρ of J/J^1 and a β-extension κ of η such that $\kappa \otimes \rho$ is contained in π. Now one has to prove:

Theorem 9.1 (Exhaustion)**.** The representation $\kappa \otimes \rho$ is a cuspidal type. In particular $\mathrm{ind}_{\mathrm{J}}^{\mathrm{G}}(\kappa \otimes \rho) \cong \pi$.

Note: The induction assertion is given by Theorem 7.3. The second main theorem is:

Theorem 9.2 (Intertwining implies conjugacy, cf. [20] 11.9 for $\mathrm{G_L}$)**.** Suppose (λ, J) and (λ', J') are two cuspidal types of G which intertwine in G (or equivalently which compactly induce equivalent representations of G.) Then there is an element $g \in \mathrm{G}$ such that $g\mathrm{J}g^{-1} = \mathrm{J}'$ and ${}^g\lambda$ is equivalent to λ'.

The proofs of those two Theorems will occupy the next two sections.

10. Exhaustion

10.1. **Bushnell–Kutzko type theory for showing non-cuspidality.** There is a well–know procedure to show exhaustion results using the computational results in [13]. We give a remark for the modular case and we exclusively use their notation in this subsection. Let here G be the set of F-rational points of a connected reductive group defined over F, and let $\mathrm{P} = \mathrm{MN}_u$ be a parabolic subgroup with opposite unipotent group N_l. The subscripts, see for example $\mathrm{J}_u, \mathrm{J}_\mathrm{M}, \mathrm{J}_l$ below, are corresponding to the Iwahori decomposition with respect to $\mathrm{N}_l\mathrm{MN}_u$. The statements and proofs of [13, 6.8–7.9(i) and 7.9(ii injectivity)] carry over to the modular case if there J_u and J_l are supposed to be pro-p, but one has to undertake two modifications:

(i) Given a representation (J, τ) on a compact open subgroup of G the formula for the convolution in the Hecke algebra $\mathcal{H}(\mathrm{G}, \tau)$ is given by
$$(\Phi \star \Psi)(x) = \sum_{[z] \in \mathrm{J}\backslash \mathrm{G}} \Phi(xz^{-1})\Psi(z),$$
to avoid any Haar measure.

(ii) The formula in [13, 6.8] needs to be replaced by:
$$(\Phi \star \Psi)(x) = [\mathrm{J}_l : \mathrm{J}_l^w] \sum_{[y_\mathrm{M}] \in \mathrm{J_M} \cap \mathrm{J}_\mathrm{M}^w \backslash \mathrm{J_M}} \varphi(xy_\mathrm{M}^{-1}w^{-1})\psi(wy_\mathrm{M})q(x, y_\mathrm{M}),$$
for $x \in \mathrm{M}$.

(Note that we did deliberately not include the surjectivity statement of [13, 7.9(ii)].) The main ingredient for the exhaustion arguments is the following.

Theorem 10.1 ([2] (0.4), [13] 7.9(ii injective) also for mod l)**.** Suppose (τ, J) is an irreducible representation (with coefficients in **C**) which decomposes under the Iwahori decomposition $\mathrm{N}_l\mathrm{M}\mathrm{N}_u$ and suppose that J_u and J_l are pro-p. Suppose further that there is a (P,J)-strongly positive element ζ in the center of M such that there is an invertible element of the Hecke algebra $\mathcal{H}(\mathrm{G},\tau)$ with support in $\mathrm{J}\zeta\mathrm{J}$. Let (π,G) be a smooth representation of G. Then there is an injective map:

$$\mathrm{Hom}_{\mathrm{J}}(\tau,\pi)\hookrightarrow\mathrm{Hom}_{\mathrm{J}_{\mathrm{M}}}(\tau,\pi_{\mathrm{N}_u}).$$

where π_{N_u} is the corresponding Jacquet module. In particular, if $\mathrm{P}\neq\mathrm{G}$ and τ is contained in π then π is not cuspidal.

10.2. **Skew characters and cuspidality.** In this section we prove:

Theorem 10.2 (cf. [41] Theorem 5.1)**.** Let π be a cuspidal irreducible representation of G and $\theta\in\mathrm{C}(\Delta)$ with $r=0$ be a self-dual semisimple character contained in π. Then Δ is skew-semisimple, i.e. the adjont involution of h acts trivially on the index set of Δ.

Proof of Theorem 10.2: Suppose for deriving a contradiction that Δ is not skew-semi-simple. Consider the decomposition

$$\mathrm{V}=\mathrm{V}_+\oplus\mathrm{V}_0\oplus\mathrm{V}_-, \tag{10.3}$$

given by $\mathrm{V}_\delta:=\oplus_{i\in\mathrm{I}_\delta}\mathrm{V}^i$, $\delta\in\{+,0,-\}$. Let M be the Levi subgroup of G defined over F given by the stabilizer of the decomposition (10.3). The decomposition also defines unipotent subgroups: N_+, the unipotent radical of the stabilizer of the flag $\mathrm{V}_+,\mathrm{V}_+\oplus\mathrm{V}_0,\mathrm{V}$ in G, and the opposite N_-. We have the Iwahori decompositions for H^1 and J^1 with respect to $\mathrm{N}_-\mathrm{M}\mathrm{N}_+$, and we write H^1_δ and J^1_δ for the obvious intersections (e.g. $\mathrm{J}^1_+:=\mathrm{J}^1\cap\mathrm{N}_+$) , $\delta\in\{\pm,0\}$. Note that every irreducible representation of $\mathrm{K}:=\mathrm{H}^1\mathrm{J}^1_+$ containing θ is a character, because K/H^1 is abelian and θ admits an extension ξ to K which is trivial on J^1_+.

Proposition 10.4. The group J^1 act transitively on the set of characters of K extending θ.

Proof. The group K is normalized by J^1 because $\mathrm{J}^1/\mathrm{H}^1$ is abelian. A character of K extending θ is contained in $\mathrm{ind}_{\mathrm{H}^1}^{\mathrm{J}^1}\theta$ and is therefore contained in η. Thus the J^1-action on the set of these characters must be transitive because η is irreducible. □

For the notion of cover we refer to [13, §8]. In fact we take the weaker version where we only want to consider strongly positive elements for the parabolic subgroup $\mathrm{Q}:=\mathrm{M}\mathrm{N}_+$, i.e. not for other parabolic subgroups. This is enough for our purposes.

Proposition 10.5. [cf.[41] Proposition 4.6 and [13] Corollary 6.6] There exists a strongly positive (Q,K)-element ζ of the centre of M, such that there is an invertible element of the Hecke algebra $\mathcal{H}(\mathrm{G},\xi)$ with support in $\mathrm{K}\zeta\mathrm{K}$.

Proof. The element ζ defined as follows (see [41, 4.5])

$$\begin{pmatrix}\varpi_{\mathrm{F}} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \varpi_{\mathrm{F}}^{-1}\end{pmatrix}$$

is central in M and a strongly (P, K)-positive element. Note that ζ and ζ^{-1} intertwine ξ, because (ξ, K) respects the Iwahori-decomposition with respect to $\mathrm{N}_-\mathrm{M}\mathrm{N}_+$. Now the argument at the end of the proof in [12, 6.6] finishes the proof, because

$$\begin{aligned} \mathrm{I}_{\mathrm{N}_+}(\xi) &\subseteq \mathrm{N}_+ \cap \mathrm{I}(\theta) \\ &= \mathrm{N}_+ \cap (\mathrm{J}^1 \mathrm{G}_\beta \mathrm{J}^1) \\ &= \mathrm{N}_+ \cap \mathrm{J}^1. \end{aligned}$$

and because the constant c there is equal to $[\mathrm{K} : \mathrm{K} \cap \zeta^{-1}\mathrm{K}\zeta]$ and therefore not divisible by $p_{\mathbf{C}}$. □

We now can finish the proof of Theorem 10.2: As π contains θ it also must contain an extension of θ to K and therefore, we obtain from Proposition 10.4 that π contains ξ. Thus π has a non-trivial Jacquet module for the parabolic Q, by Proposition 10.5 and Theorem 10.1. A contradiction, because $\mathrm{M} \neq \mathrm{G}$ and π is cuspidal. This finishes the proof of Theorem 10.2.

10.3. **Exhaustion of cuspidal types (Proof of Theorem 9.1).** The proof of exhaustion is mutatis mutandis to [42] (and to [21] for mod-$p_{\mathbf{C}}$), see also [24, 3.3] for the final argument. We are going to give the outlook of the proof in this section and refer to the corresponding results in [42]. The referred statements of [42, §6 and 7] are mutatis mutandis valid for the quaternionic case. To start, let π be a cuspidal irreducible representation of G. Then, by Theorem 3.1 and Theorem 10.2, there exists a skew-semisimple character $\theta \in \mathrm{C}(\Lambda, 0, \beta)$ such that:

$$\theta \subseteq \pi. \tag{10.6}$$

Let Λ be chosen such that $\tilde{\mathfrak{b}}(\Lambda)$ is minimal with respect to (10.6). The aim is to show that $\mathrm{P}^0(\Lambda_\beta)$ is maximal parahoric. Take any β-extension $(\kappa, \mathrm{J}^0_\Lambda)$ of θ (with respect to some maximal $\tilde{\mathfrak{b}}(\Lambda_\mathrm{M})$ containing $\tilde{\mathfrak{b}}(\Lambda)$, see §6.5). Then, by Lemma 6.5(ii) there is an irreducible representation ρ of $\mathrm{P}^0(\Lambda_\beta)/\mathrm{P}_1(\Lambda_\beta)$ such that $\lambda := \kappa \otimes \rho$ is contained in π. Note that ρ has to be cuspidal by the minimality of $\tilde{\mathfrak{b}}(\Lambda)$, see [42, 7.4] and use Proposition 6.4(ii) instead of [42, Lemma 7.5].

Further by the minimality condition on $\tilde{\mathfrak{b}}(\Lambda)$ there is a tuple of idempotents $(e_j)_{j=-m}^m$ exactly subordinate to $\Delta = [\Lambda, -, 0, \beta]$ such that $\mathrm{P}^0(\Lambda^{(0)}_\beta)$ is a maximal parahoric of $\mathrm{G}^{(0)}_\beta$ if e_0 is non-zero. (by [42, 7.7]; take quotient instead of subrepresentation in the definition of *lying over*). Indeed: There exists a tuple of lattice functions $(\Gamma^{(i,0)})_{i\in\mathrm{I}} \in \prod_{i\in\mathrm{I}} \mathrm{Latt}^1_{h_i,\mathfrak{o}_{\mathrm{E}_i},\mathfrak{o}_\mathrm{D}}(\mathrm{W}^{(0)} \cap \mathrm{V}^i)$ such that $\tilde{\mathfrak{b}}(\Gamma^{(0)}) \subsetneq \tilde{\mathfrak{b}}(\Lambda^{(0)})$ and $\mathrm{P}^0(\Gamma^{(0)}_\beta) = \mathrm{P}^0(\Lambda^{(0)}_\beta)$, for $\Gamma^{(0)} := \oplus_{i\in\mathrm{I}}\Gamma^{(i,0)}$, if $\mathrm{P}^0(\Lambda^{(0)}_\beta)$ is not a maximal parahoric subgroup of $\mathrm{G}^{(0)}_\beta$, by Lemma C.1. Now take a self-dual $\mathfrak{o}_\mathrm{E}$-$\mathfrak{o}_\mathrm{D}$-lattice sequence Λ' split by $(e^{(j)})_j$ such that $\oplus_{j\neq 0}\Lambda'^{(j)}$ is an affine translation of $\oplus_{j\neq 0}\Lambda^{(j)}$, $\tilde{\mathfrak{b}}(\Lambda')$ is contained in $\tilde{\mathfrak{b}}(\Lambda)$ and $\tilde{\mathfrak{b}}(\Gamma^{(0)}) = \tilde{\mathfrak{b}}(\Lambda'^{(0)})$. We then apply [42, Lemma 7.7] to conclude that the transfer of θ to $\mathrm{C}(\Lambda', 0, \beta)$ is contained in π. A contradiction, because $\tilde{\mathfrak{b}}(\Lambda'^{(0)})$ is properly contained in $\tilde{\mathfrak{b}}(\Lambda^{(0)})$.

Assume $\mathrm{P}^0(\Lambda_\beta)$ is not a maximal parahoric of G_β (Note that in our quaternionic case the center of G_β is compact, i.e. $\mathrm{SO}(1,1)(\mathrm{F})$ does not occur as a factor of G_β). We then have $m>0$, i.e. $(e_j)_{j=-m}^m$ has at least 2 idempotents. Let M be the stabilizer in G of the decomposition of V given by $(e_j)_{j=-m}^m$, and let U be the set of upper unipotent elements of G with respect to the latter decomposition. We put $\mathrm{P}=\mathrm{M}\,\mathrm{U}$. Let λ_{P} be the natural representation of $\mathrm{J}_{\mathrm{P}}^0=\mathrm{H}_\Lambda^1(\mathrm{J}_\Lambda^0\cap\mathrm{P})$ on the set of $(\mathrm{U}\cap\mathrm{J}_\Lambda^0)$-fixed points of λ.

We need more notation corresponding to the splitting $(e_j)_{j=-m}^m$:

(i) For every non-zero j there is a unique $i_j\in\mathrm{I}$ such that $e_j\beta_{i_j}\neq 0$. Under the map j_{E} the o_{E}-$\mathfrak{o}_D$-lattice sequence corresponds to a tuple $(\Lambda_\beta^i)_{i\in\mathrm{I}}$, with Λ_β^i a self-dual $\mathfrak{o}_{\mathrm{D}_\beta^i}$-lattice sequence in V_β^i. Further V_β^i and Λ_β^i decompose under $(e_j)_{j=-m}^m$:

$$\mathrm{V}_\beta^i=(\bigoplus_{j\neq 0,i_j=i}\mathrm{V}_\beta^{(j)})\oplus\mathrm{V}_\beta^{(i,0)},\ \Lambda_\beta^i=(\bigoplus_{j\neq 0,i_j=i}\Lambda_\beta^{(j)})\oplus\Lambda_\beta^{(i,0)}$$

($\mathrm{V}_\beta^{(i,0)}=0$ is possible). Then

$$\mathrm{P}^0(\Lambda_\beta)/\mathrm{P}_1(\Lambda_\beta)\simeq\bigtimes_{j=0}^m\mathrm{P}^0(\Lambda_\beta^{(j)})/\mathrm{P}_1(\Lambda_\beta^{(j)}),\tag{10.7}$$

because the $\mathrm{E}\otimes_{\mathrm{F}}\mathrm{D}$-partition $(e_j)_{j=-m}^m$ is exactly subordinate to Δ. Without loss of generality we assume that Λ_β^i is *standard*, i.e.

$$2|e_i=e(\Lambda_\beta^i|o_{\mathrm{D}_\beta^i})\text{ and }\Lambda_\beta^i(t)^{\#_i}=\Lambda_\beta^i(1-t),$$

for all $t\in\mathbb{Z}$, where $\#_i$ is the duality operator defined by h_β^i.

(ii) For every $j\neq 0$ there is a unique integer q_j satisfying $-\frac{e_{i_j}}{2}<q_j<\frac{e_{i_j}}{2}$ such that

$$\Lambda_\beta^{(j)}(q_j)\supsetneq\Lambda_\beta^{(j)}(q_j+1),$$

and if we fix a total order on the index set I of β then we choose the numbering of the idempotents e_j the way such that we have for non-zero j,k:

$$j<k\text{ if and only if }(i_j<i_k\text{ or }(i_j=i_k\text{ and }q_j<q_k)),\tag{10.8}$$

see [42, §6.2 and remark after Lemma 6.6].

(iii) For every positive j Stevens constructs two Weyl group elements s_j and s_j^{ϖ} of G_β, which through conjugation swap the blocks corresponding to j and $-j$ and act trivially on $\mathrm{W}^{(k)}$ for $k\neq\pm j$, see [42, §6.2]. Now they are used to define an involution on $\mathrm{Aut}_{\mathrm{D}}(\mathrm{W}^{(j)})$ via:

$$\sigma_j(g_j)=s_j\sigma_h(g_j)^{-1}s_j^{-1}.$$

The representation ρ decomposes under 10.7 into

$$\rho\simeq\rho^{(0)}\otimes(\bigotimes_{j=1}^m\tilde{\rho}^{(j)}),$$

and one defines for negative j:

$$\tilde{\rho}^{(-j)}=\tilde{\rho}^{(j)}\circ\sigma_j.$$

We have now two cases to consider:

Case 1: There exists a positive j such that $\rho^{(j)} \not\simeq \rho^{(-j)}$. In this case Stevens constructs in [42, 7.2.1] a decomposition $\mathrm{Y}_{-1} \oplus \mathrm{Y}_0 \oplus \mathrm{Y}_1$ with Levi M' (the stabilizer of the decomposition) and non-zero Y_{-1}, Y_1 such that the normalizer of $\rho|_{\mathrm{M}\cap\mathrm{P}^0(\Lambda_\beta)}$ in G_β is contained in M'. Then by [13, Theorem 7.2] and [42, 6.16] (see [21, Lemma 9.8] for the modular case) the representation λ_{P} satisfies the conditions of Theorem 10.1, for a parabolic of G with levi M'.

Case 2: For all positive j we have $\rho^{(j)} \simeq \rho^{(-j)}$. Here let

$$\mathrm{Y}_{-1} := e^{(-m)}\mathrm{V},\ \mathrm{Y}_0 := (1 - e^{(m)} - e^{(-m)})\mathrm{V},\ \mathrm{Y}_1 := e^{(m)}\mathrm{V}$$

with stabilizer M' in G and (upper block triangular) parabolic P'. Here Stevens constructs in [42, 7.2.2] a strongly $(\mathrm{P}', \mathrm{J}^0_{\mathrm{P}})$-positive element ζ of G in the centre of M' such that there is an invertible element of $\mathcal{H}(\mathrm{G}, \lambda_{\mathrm{P}})$ with support in $\mathrm{J}^0_{\mathrm{P}}\zeta\mathrm{J}^0_{\mathrm{P}}$. The construction of ζ carries mutatis mutandis over to the quaternionic case using the ordering (10.8). The elements s_m and s_m^{ϖ} in Section *loc.cit.* are automatically in G because all isometries of h have reduced norm 1, so one does not need to consider *loc.cit.* (7.2.2)((i) and (ii)). Further, for the modular case, in the paragraph after [42, 7.12], as indicated in [21, Theorem 9.9(ii)], one needs to refer to the description of the Hecke algebra of a cuspidal representation on a maximal parahoric given by Geck–Hiss–Malle [17, 4.2.12]. Thus λ_{P} satisfies the conditions of Theorem 10.1 with respect to P'.

In either case Theorem 10.1 and the fact that $\mathrm{M}' \cap \mathrm{G}$ is a proper Levi subgroup of G imply that π is not cuspidal. A contradiction.

11. Conjugate cuspidal types (Proof of Theorem 9.2)

In this section we finish the classification of cuspidal irreducible representations of G. Recall from the assumption of Theorem 9.2 that we are given two cuspidal types $(\lambda, \mathrm{J}(\beta, \Lambda))$ and $(\lambda', \mathrm{J}(\beta', \Lambda'))$ which induce equivalent representations of G. Let us denote the representation $\mathrm{ind}_{\mathrm{J}}^{\mathrm{G}} \lambda$ by π. Let $\theta \in \mathrm{C}(\Lambda, 0, \beta)$ and $\theta' \in \mathrm{C}(\Lambda', 0, \beta')$ be the skew-semisimple characters used for the construction of λ and λ'. As θ and θ' are contained in the irreducible π we obtain that both have to intertwine by an element of G, say with matching $\zeta : \mathrm{I} \to \mathrm{I}'$. By **II**.6.10, cf. [20, Proposition 11.7], we can assume without loss of generality that β and β' have the same characteristic polynomial and that θ' is the transfer of θ from (β, Λ) to (β', Λ'). One can now apply a †-construction, see **I**.§5.3 to transfer to the case where all V^i and $\mathrm{V}^{i'}$ have the same D-dimension, to then observe that the matching ζ must fulfill that β_i and $\beta'_{\zeta(i)}$ have the same minimal polynomial by the unicity of the matching. We can therefore conjugate to the case $\beta = \beta'$, by **II**.4.14. Now, [21, Theorem 12.3] is valid for the quaternionic case, see below. We conclude that there is an element g of G such that $g\mathrm{J}g^{-1} = \mathrm{J}'$ and ${}^g\lambda \sim \lambda'$.

Let us outline *loc.cit.* to show which of their statements and constructions are needed: Without loss of generality we can assume that Λ and Λ' are standard self-dual with the same o_{D}-period. We consider the associated self-dual o_{E}-o_{D}-lattice functions Γ and Γ' respectively and, in the centralizer, $\Gamma_\beta = j_\beta(\Gamma)$ and $\Gamma'_\beta = j_\beta(\Gamma')$. λ is constructed using an irreducible representation ρ of $\mathrm{M}(\Gamma_\beta) := \mathbb{P}(\Gamma_\beta)(\mathrm{k_F})$ with cuspidal restriction to $\mathrm{M}^0(\Gamma_\beta) :=$

$\mathbb{P}^0(\Gamma_\beta)(\mathrm{k_F})$ and a β-extension (κ, J), $\lambda = \kappa \otimes \rho$. Analogously we have $\lambda' = \kappa' \otimes \rho'$ for respective ρ' and κ'. We now have three pairs of functors:

(i) $\mathrm{R}_\kappa : \mathfrak{R}(\mathrm{G}) \to \mathfrak{R}(\mathrm{M}(\Gamma_\beta))$ and $\mathrm{I}_\kappa : \mathfrak{R}(\mathrm{M}(\Gamma_\beta)) \to \mathfrak{R}(\mathrm{G})$ defined via

$$\mathrm{R}_\kappa(\omega) := \mathrm{Hom}_{\mathrm{J}^1}(\kappa, \omega),\ \mathrm{I}_\kappa(\varrho) := \mathrm{ind}_{\mathrm{J}}^{\mathrm{G}}(\kappa \otimes \varrho)$$

(ii) $\mathrm{R}_{\Gamma_\beta} : \mathfrak{R}(\mathrm{G}_\beta) \to \mathfrak{R}(\mathrm{M}(\Gamma_\beta))$ and $\mathrm{I}_{\Gamma_\beta} : \mathfrak{R}(\mathrm{M}(\Gamma_\beta)) \to \mathfrak{R}(\mathrm{G}_\beta)$ defined via

$$\mathrm{R}_{\Gamma_\beta}(\omega) := \mathrm{Hom}_{\mathrm{P}_1(\Gamma_\beta)}(1, \omega),\ \mathrm{I}_{\Gamma_\beta}(\varrho) := \mathrm{ind}_{\mathrm{P}(\Gamma_\beta)}^{\mathrm{G}_\beta}(\varrho)$$

(iii) $\mathrm{R}^0_{\Gamma_\beta} : \mathfrak{R}(\mathrm{G}_\beta) \to \mathfrak{R}(\mathrm{M}^0(\Gamma_\beta))$ and $\mathrm{I}^0_{\Gamma_\beta} : \mathfrak{R}(\mathrm{M}^0(\Gamma_\beta)) \to \mathfrak{R}(\mathrm{G}_\beta)$ defined via

$$\mathrm{R}^0_{\Gamma_\beta}(\omega) := \mathrm{Hom}_{\mathrm{P}_1(\Gamma_\beta)}(1, \omega),\ \mathrm{I}^0_{\Gamma_\beta}(\varrho) := \mathrm{ind}_{\mathrm{P}^0(\Gamma_\beta)}^{\mathrm{G}_\beta}(\varrho).$$

Before we start to explain their proof we want to remark that we use [21, 8.5] which is valid for the case of G, because the key is the exact diagram in the proof of [21, Lemma 5.2] which can be obtained for G by taking Gal(L|F)-fixed points of the corresponding diagram for $\mathrm{G} \otimes \mathrm{L}$. Now we come to their proof of [21, Theorem 12.3]. It contains two parts:

Part 1: The first part is to show that Γ and Γ' are G_β-conjugate (up to affine translation). It is implied as follows: We have that $\mathrm{I}_\kappa(\rho)$ and $\mathrm{I}_{\kappa'}(\rho')$ are isomorphic to π, in particular isomorphic to each other, and therefore $\mathrm{R}_\kappa \circ \mathrm{I}_{\kappa'}(\rho')$ contains ρ and therefore is non-zero. Thus $\mathrm{R}_{\Gamma_\beta} \circ \mathrm{I}_{\Gamma'_\beta}(\rho')$ is non-zero by [21, 8.5(i)]. Let ρ'^0 be a cuspidal irreducible sub-representation of the restriction of ρ' to $\mathrm{M}^0(\Gamma'_\beta)$. Then $\mathrm{R}_{\Gamma_\beta} \circ \mathrm{I}^0_{\Gamma'_\beta}(\rho'^0)$ is non-zero because it contains $\mathrm{R}_{\Gamma_\beta} \circ \mathrm{I}_{\Gamma'_\beta}(\rho')$. Thus $\mathrm{R}^0_{\Gamma_\beta} \circ \mathrm{I}^0_{\Gamma'_\beta}(\rho'^0)$ is non-zero and therefore $\mathrm{P}^0(\Gamma_\beta)$ and $\mathrm{P}^0(\Gamma'_\beta)$ are G_β-conjugate by [21, 7.2(ii)]. Therefore Γ_β is G_β-conjugate to Γ'_β (up to affine translation) because $\mathrm{P}^0(\Gamma_\beta)$ and $\mathrm{P}^0(\Gamma'_\beta)$ are G_β-conjugate maximal parahoric subgroups of G_β, see [7, after 5.2.6]. This finishes Part 1 and we can assume $\Lambda = \Lambda'$ without loss of generality, in particular $\theta = \theta'$ and we have the same Heisenberg representation.

Part 2 is for showing $\lambda \simeq \lambda'$. Take a character χ of $\mathrm{M}(\Lambda_E)$ such that $\kappa' = \kappa \otimes \chi$. Then we get $\lambda' = \kappa \otimes (\chi \otimes \rho')$ and we get

$$(\mathrm{R}_\kappa \circ \mathrm{I}_\kappa)(\chi \otimes \rho') \cong (\mathrm{R}_\kappa \circ \mathrm{I}_\kappa)(\rho)$$

where the left hand side contains $\chi \otimes \rho'$ and the right hand side is equivalent to ρ by [21, 8.5(ii)]. Thus by irreducibility we obtain the existence of an isomorphism from $\chi \otimes \rho'$ to ρ, and therefore of an isomorphism from λ' to λ.

Appendix A. Erratum on semimsimple strata for p-adic classical groups

This part of the appendix is a fix of [40, Proposition 4.2]. As it is stated in *loc.cit.* the proposition is false. This was pointed out by Blondel and Van-Dinh Ngo. The fix was provided by Stevens, author of [42], in 2012, but until now not published. At first we need to set up the notation to discuss the proposition. In Appendix A and B we only work over F (not D) and, as usual, with odd residual characteristic p, and we are given an ϵ-hermitian form (h, V) with respect to some at most quadratic extension $\mathrm{F}|\mathrm{F}_0$ with Galois group $\langle \bar{\ } \rangle$. We fix a uniformizer ϖ of F with $\varpi \in \mathrm{F}_0$ if $\mathrm{F}|\mathrm{F}_0$ is unramified and $\sigma_h(\varpi) = -\varpi$ if $\mathrm{F}|\mathrm{F}_0$ is ramified. Further we will use a fixed uniformizer ϖ_0 of F_0 which satisfies $\varpi = \varpi_0$ if $\mathrm{F}|\mathrm{F}_0$ is unramified and $\varpi_0 = \varpi^2$ if not. Given a lattice sequence Λ on V Stevens defines a finite dimensional $\mathrm{k_F}$-vector space $\tilde{\Lambda}$ via

$$\tilde{\Lambda} := \tilde{\Lambda}(0) \oplus \tilde{\Lambda}(1) \oplus \ldots \oplus \tilde{\Lambda}(e_0 - 1),\ \tilde{\Lambda}(j) = \Lambda(j)/\Lambda(j+1),\ j \in \mathbb{Z},$$

where e_0 is the F_0-period of Λ and considers its endomorphism algebra $\mathrm{End}_{\mathrm{k_F}}(\tilde{\Lambda})$. The space $\tilde{\Lambda}(j)$ is identified with $\tilde{\Lambda}(j + ie_0)$ by multiplication with the ith power of ϖ_0, so instead of $\tilde{\Lambda}(j)$ we write $\tilde{\Lambda}(\tilde{j})$, $\tilde{j}$ being the mod e_0 congruence class of j. We consider a stratum $\Delta = [\Lambda, n, n-1, b]$. One defines an endomorphism $\tilde{b} \in \mathrm{End}_{\mathrm{k_F}}(\tilde{\Lambda})$ by the maps

$$\tilde{\Lambda}(\tilde{j}) \to \tilde{\Lambda}(\tilde{j} - \tilde{n}),\ [v] \mapsto [bv],\ \tilde{j} \in \mathbb{Z}/e_0\mathbb{Z}.$$

Now, Proposition 4.2 in *loc.cit.* states that if $[\Lambda, n, n-1, b]$ is a self-dual stratum then there is a self-dual stratum $[\Lambda', n', n'-1, b']$ with $b' = b$ such that

$$\frac{n'}{e(\Lambda'|\mathrm{F})} = \frac{n}{e(\Lambda|\mathrm{F})},\ \tilde{\mathfrak{a}}_{1-n}(\Lambda) \subseteq \tilde{\mathfrak{a}}_{1-n'}(\Lambda')$$

with semisimple endomorphism $\tilde{b'}$ in $\mathrm{End}_{\mathrm{k_F}}(\widetilde{\Lambda'})$. Here is a counter example. We take the symplectic group $\mathrm{Sp}_6(\mathbb{Q}_3)$ with the usual anti-diagonal Gram matrix and we consider the self-dual lattice chain Λ to the following hereditary order together with an element b:

$$\begin{pmatrix} \mathfrak{o} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{pmatrix},\ \begin{pmatrix} & & -\varpi & & & \\ -1 & & & & & \\ & -1 & & & & \\ & & & & & \varpi \\ & & & 1 & & \\ & & & & 1 & \end{pmatrix}\varpi^{-1},$$

e.g. take $\varpi = 3$. The stratum $\Delta = [\Lambda, 2, 1, b]$ is fundamental with characteristic polynomial $\chi_\Delta(X) = (X-1)^3(X+1)^3$. Note that Λ is a regular lattice chain of period 3. Now suppose $\Delta' = [\Lambda', n', n'-1, b' = b]$ is another stratum not equivalent to a null stratum such that Δ has the same depth as Δ', i.e. $\frac{n}{e} = \frac{n'}{e'}$ for $e = e(\Lambda|\mathrm{F})$ and $e' = e(\Lambda'|\mathrm{F})$. The equality of depth and $b = b'$ imply that Δ and Δ' are intertwining fundamental strata which share the characteristic polynomial. From $\chi_{\Delta'}(0) = -1 \in \mathrm{k}_\mathrm{F}^\times$ follows now that b' normalizes Λ'. Now if $\tilde{b'} \in \mathrm{End}_{\mathrm{k_F}}(\tilde{\Lambda'})$ is semisimple its minimal polynomial has to divide $(X-1)(X+1)$ because its third power vanishes $\widetilde{b'}$ (Note that 3 is the residual characteristic). In other words: $X+1$ and $X(X+1)$ define the same endomorphism for $X = \tilde{b'}$. Thus by homogeneity e' divides n' or $2n'$, i.e. $\frac{2n'}{e'}$ is an integer, which is absurd, as $e = 3$ and $n = 2$. We were able to exclude Δ' equivalent to a null stratum immediately by [36, Proposition 6.9]. Given a lattice sequence Λ we call another lattice sequence Λ' a refinement of Λ if there is positive integer m such that $\Lambda'(mj) = \Lambda(j)$ for all $j \in \mathbb{Z}$. Now, Proposition 4.2 in *loc.cit.* will be replaced by:

Proposition A.1 (S. Stevens, 2012)**.** Let $\Delta = [\Lambda, n, n-1, b]$ be a self-dual stratum. Then there is a self-dual stratum $\Delta' = [\Lambda', n', n'-1, b' = b]$ such that:

(i) $\tilde{\mathfrak{a}}_{1-n}(\Lambda) \subseteq \tilde{\mathfrak{a}}_{1-n'}(\Lambda')$ and Λ' is a refinement of Λ,

(ii) $\frac{n}{e} = \frac{n'}{e'}$ with $e = e(\Lambda|\mathrm{F})$ and $e' = e(\Lambda'|\mathrm{F})$,

(iii) and if we define $y = \varpi^{n/g} b^{e/g}$ with $g = \gcd(e, n)$ we have $\tilde{y} \in \mathrm{End}_{\mathrm{k_F}}(\widetilde{\Lambda'})$ is semisimple.

(iv) If Δ is non-fundamental then we can choose Δ' such that it further satisfies $\tilde{b} = 0$ in $\mathrm{End}_{\mathrm{k_F}}(\widetilde{\Lambda'})$, i.e. such that Δ' is equivalent to a null stratum.

Proof by S. Stevens, edited by the author. In passing through the affine class of Λ we can assume Λ to be standard self-dual without loss of generality. Denote by $\Phi(X) \in \mathfrak{o}_{\mathrm{F}}[X]$ the characteristic polynomial of y and by $\varphi(X) \in \mathrm{k_F}[X]$ its reduction modulo $\mathfrak{p}_{\mathrm{F}}$. Then by the definition of y there is a sign η which satisfies $\sigma_h(y) = \eta y$ and $\overline{\Phi}(\eta X) = \pm\Phi(X)$ and the same applies to φ. We write

$$\varphi(X) = \psi_0(X)\psi_1(X),$$

with $\psi_0(X)$ a power of X and coprime to $\psi_1(X)$. By Hensel's Lemma, this lifts to a factorization

$$\Phi(X) = \Psi_0(X)\Psi_1(X),$$

such that $\overline{\Psi}_i(\eta X) = \pm\Psi_i(X)$. For $i = 0, 1$ we put $\Upsilon_i = \ker \Psi_i(y)$, so that $\mathrm{V} = \Upsilon_0 \perp \Upsilon_1$ and this decomposition is stabilized by b. Moreover, putting $\Lambda_i(k) = \Lambda(k) \cap \Upsilon_i$, we have $\Lambda(k) = \Lambda_0(k) \oplus \Lambda_1(k)$ for all $k \in \mathbb{Z}$. Now we pass to the graded $\mathrm{k_F}$-vector space $\tilde{\Lambda}$ write $\mathcal{Y}_i$ for the image in $\tilde{\Lambda}$ of the space Υ_i; that is

$$\mathcal{Y}_i = \bigoplus_{k=0}^{e_0-1} (\Lambda(k) \cap \Upsilon_i) / (\Lambda(k+1) \cap \Upsilon_i)\,.$$

Then we have an orthogonal decomposition $\tilde{\Lambda} = \mathcal{Y}_0 \perp \mathcal{Y}_1$, with respect to the (graded) nondegenerate ε-hermitian pairing $\tilde{h}$ on $\tilde{\Lambda}$, see [40, §4] where $\tilde{h}$ is defined using that Λ is standard self-dual. Both b and y induce homogeneous graded maps $\tilde{b}$ and $\tilde{y}$ in $\mathrm{End}_{\mathrm{k_F}}(\tilde{\Lambda})$, respectively of degree $-n$ and 0 mod $e_0\mathbb{Z}$. Then we can also interpret $\mathcal{Y}_i$ as the kernel of the map $\psi_i(\tilde{y})$ (which is also homogeneous of degree 0) and b preserves the decomposition $\tilde{\Lambda} = \mathcal{Y}_0 \perp \mathcal{Y}_1$. The restriction of $\tilde{b}$ to $\mathcal{Y}_0$ is nilpotent, since $\tilde{y}^2 = \tilde{b}^{2e/g}$ is nilpotent on $\mathcal{Y}_0$. On the other hand, the restriction of $\tilde{y}$ to $\mathcal{Y}_1$ is invertible and has a Jordan decomposition

$$\tilde{y}|_{\mathcal{Y}_1} = \tilde{y}_s + \tilde{y}_n,$$

with both $\tilde{y}_s, \tilde{y}_n$ homogeneous of degree 0. Note that $\tilde{y}|_{\mathcal{Y}_1}$ can be written as a polynomial in $\tilde{y}$ so the same applies to $\tilde{y}_n$ and $\tilde{y}_s$; in particular, they commute with $\tilde{b}$. We pick an odd integer $m = 2s - 1$ such that both $\tilde{y}_n^m = 0$ and $\tilde{b}^m|_{\mathcal{Y}_0} = 0$. Now we put

$$\mathcal{V}^i_{\tilde{k}} = \tilde{b}^i\left(\tilde{\Lambda}(\tilde{k} + i\tilde{n}) \cap \mathcal{Y}_0\right) \perp \tilde{y}_n^i\left(\tilde{\Lambda}(\tilde{k}) \cap \mathcal{Y}_1\right), \qquad \text{for } \tilde{k} \in \mathbb{Z}/e_0\mathbb{Z},\ 0 \le i \le m;$$

$$\mathcal{W}^i_{\tilde{k}} = \bigcap_{q-p=2(s-1-i)} \left(\mathcal{V}^p_{\tilde{k}} + \left(\mathcal{V}^q_{-\tilde{k}}\right)^{\perp}\right), \qquad \text{for } \tilde{k} \in \mathbb{Z}/e_0\mathbb{Z},\ 0 \le i \le s-1;$$

$$\mathcal{W}^i_{\tilde{k}} = \left(\mathcal{W}^{m-i}_{-\tilde{k}}\right)^{\perp}, \qquad \text{for } \tilde{k} \in \mathbb{Z}/e_0\mathbb{Z},\ s \le i \le m.$$

Note that we have $\mathcal{W}^i_{\tilde{k}} = \left(\mathcal{W}^i_{\tilde{k}} \cap \mathcal{Y}_0\right) \perp \left(\mathcal{W}^i_{\tilde{k}} \cap \mathcal{Y}_1\right)$ and $\tilde{b}\left(\mathcal{W}^i_{\tilde{k}} \cap \mathcal{Y}_0\right) \subseteq \left(\mathcal{W}^{i+1}_{\tilde{k}-\tilde{n}} \cap \mathcal{Y}_0\right)$; similarly, $\tilde{y}_n\left(\mathcal{W}^i_{\tilde{k}} \cap \mathcal{Y}_1\right) \subseteq \left(\mathcal{W}^{i+1}_{\tilde{k}} \cap \mathcal{Y}_1\right)$. Moreover,

$$\tilde{\Lambda}(\tilde{k}) = \mathcal{W}^0_{\tilde{k}} \supseteq \mathcal{W}^1_{\tilde{k}} \supseteq \cdots \supseteq \mathcal{W}^m_{\tilde{k}} = 0, \quad \text{for } \tilde{k} \in \mathbb{Z}/e_0\mathbb{Z},$$

and $\tilde{\varpi}_{\mathrm{F}}\mathcal{W}^i_{\tilde{k}} = \mathcal{W}^i_{\tilde{k}+\tilde{e}}$, for $\tilde{k} \in \mathbb{Z}/e_0\mathbb{Z}$, $0 \le i \le m$. In particular, this gives rise to a refinement Λ' of Λ, with

$$\Lambda'(km+i)/\Lambda(k+1) = \mathcal{W}^i_{\tilde{k}}, \qquad \text{for } k \in \mathbb{Z},\ 0 \le i \le m-1.$$

We put $n' = nm$. The proof is now the same as in [40, Proposition 4.2], but with showing that $\tilde{y}$ is semisimple in $\mathrm{End}_{\mathrm{k_F}}(\widetilde{\Lambda'})$ instead of $\tilde{b}$ because of the change of statement in A.1(iii) compared to *loc.cit.*. Note that the construction implies that $y|_{\Upsilon_0}$ induces the zero map in $\mathrm{End}_{\mathrm{k_F}}(\widetilde{\Lambda'})$ so one needs only to prove that $y|_{\Upsilon_1}$ induces a semisimple map. The assertion (iv) follows now from the construction, as $\tilde{b}|_{\mathcal{Y}_0}$ is null in $\mathrm{End}_{\mathrm{k_F}}(\widetilde{\Lambda'})$. □

Now we state Theorem 4.4 of *loc.cit.*, We write G for $\mathrm{U}(h)$. For the notion of G-split stratum see [40, §2].

Theorem A.2 (cf. [40] Theorem 4.4)**.** Suppose we are given a non-G-split self-dual fundamental stratum $\Delta = [\Lambda, n, n-1, b]$ over F. Then there is a skew-semisimple stratum $\Delta' = [\Lambda', n', n'-1, \beta']$ of the same depth as Δ such that

$$b + \tilde{\mathfrak{a}}_{1-n}(\Lambda) \subseteq \beta' + \tilde{\mathfrak{a}}_{1-n'}(\Lambda'). \tag{A.3}$$

Proof by S. Stevens, edited by the author: The theorem follows as in the proof given in *loc.cit.* with the following changes. The proof there uses the fact that, a non-split fundamental stratum $[\Lambda, n, n-1, b]$ with Λ strict is equivalent to a simple stratum if and only if $\tilde{y}$ is semisimple in $\mathrm{End}_{\mathrm{k_F}}(\tilde{\Lambda})$: this is true for strata in γ-standard form by [11, Proposition 2.5.8], and follows for all strata by [11, Proposition 2.5.11]. Moreover, the same result is true for non-strict Λ since one can replace Λ by the underlying lattice chain without changing the coset or the induced map $\tilde{y}$ (cf. [6, §2.1]). □

This finishes the erratum.

Appendix B. The self-dual and the non-fundamental case

The strategy of the proof of A.2 carries to complementary general cases. This all is not new, but we need it for the main part of the article and it follows directly from §A. So we state it and give the short argument.

Theorem B.1. (i) Suppose we are given a (self-dual) fundamental stratum $\Delta = [\Lambda, n, n-1, b]$ over F. Then there is a (self-dual) semisimple stratum $\Delta' = [\Lambda', n', n'-1, \beta']$ of the same depth as Δ which satisfies (A.3).

(ii) Suppose $\Delta = [\Lambda, n, n-1, b]$ is a non-fundamental (self-dual) stratum. Then there is a (self-dual) null-stratum $\Delta' = [\Lambda', n'-1, n'-1, 0]$ such that

$$b + \mathfrak{a}_{1-n}(\Lambda) \subseteq \mathfrak{a}_{1-n'}(\Lambda')$$

and

$$\frac{n}{e(\Lambda|\mathrm{F})} = \frac{n'}{e(\Lambda'|\mathrm{F})}.$$

Proof. For the general fundamental case the proof in Theorem A.2 simplifies because one does not need orthogonal sums and does not need to consider [38, 1.10]. So let us consider the self-dual fundamental case. By the general fundamental case we can find a semisimple stratum Δ' of the same depth as Δ satisfying the condition (A.3), but with the self-dual lattice sequence Λ' constructed in Proposition A.1. Now Δ' is in addition self-dual because b is. It is equivalent to a self-dual semisimple stratum by [20, Proposition A.9].

In the second part we use that in the proof of A.1 the lattice sequence Λ' is constructed such that $\widetilde{b} \in \mathrm{End}_{\mathrm{k_F}}(\widetilde{\Lambda}')$ is 0. This finishes the proof. □

Appendix C. Non-parahoric subgroups on maximal self-dual orders

This is about a technical step for going from a vertex in the weak structure of the Bruhat-Tits building of a classical group to a vertex which supports a maximal parahoric subgroup. Here we also consider non-quaternionic classical groups. Let $h : \mathrm{V}\times\mathrm{V} \to \mathrm{D}$ be an ϵ-hermitian form with respect to $(\mathrm{D}, (\bar{\ }))$, where D is a skew-field of degree at most 2 over F and $(\bar{\ })$ is an anti-involution on D. We write G for the set of isometries of h whom if h is orthogonal we further require to have reduced norm 1. We call a hereditary order of $\mathrm{End}_{\mathrm{D}}(\mathrm{V})$ self-dual if it is stable under the adjoint anti-involution σ_h of h, and those self-dual hereditary orders which are among all self-dual hereditary orders maximal under inclusion are called *maximal self-dual orders.* We fix a uniformizer ϖ_{D} of D such that $\bar{\varpi}_{\mathrm{D}} = \pm\varpi_{\mathrm{D}}$.

Lemma C.1. Let $\Gamma \in \mathrm{Latt}^1_h(\mathrm{V})$ be a self-dual $\mathfrak{o}_{\mathrm{D}}$-lattice function such that $\tilde{\mathfrak{a}}(\Gamma)$ is a maximal self-dual order and $\mathrm{P}^0(\Gamma)$ is not a maximal parahoric subgroup of G. Then, there exists a self-dual $\mathfrak{o}_{\mathrm{D}}$-lattice function Γ', such that $\mathrm{P}^0(\Gamma') = \mathrm{P}^0(\Gamma)$ and $\tilde{\mathfrak{a}}(\Gamma') \subsetneq \tilde{\mathfrak{a}}(\Gamma)$.

Proof. Instead with Γ we work with a self-dual lattice chain $\Lambda = \Lambda_\Gamma$ associated to Γ. Because of the assumptions on Γ, after possibly multiplying h with ϖ_{D}^{-1}, we can assume without loss of generality,

- $\Lambda_0^{\#} = \Lambda_1$
- $(\bar{\ })$ is trivial on $\mathrm{k_D}$, and
- that the $\mathrm{k_D}$-form $(\bar{h}, \Lambda_0/\Lambda_1)$, given by $\bar{h}(\bar{v},\bar{w}) := \overline{h(v,w)}$, is orthogonal of the form $\mathrm{O}(1,1)$.

Now we choose a lattice $\Lambda_{\frac{1}{2}}$ in between Λ_0 and Λ_1 such that $\Lambda_{\frac{1}{2}}/\Lambda_1$ is isotropic. There are exactly 2 choices. Now we add all the $\mathrm{D}^\times$-translates of $\Lambda_{\frac{1}{2}}$ to Λ to obtain a new self-dual lattice sequence Λ' which satisfies $\tilde{\mathfrak{a}}(\Lambda') \subsetneq \tilde{\mathfrak{a}}(\Lambda)$. We claim that $\mathrm{P}^0(\Lambda')$ and $\mathrm{P}^0(\Lambda)$ coincide.

Proof of the claim in several steps:

1) At first we show that $\mathrm{P}^0(\Lambda)$ is contained in $\mathrm{P}(\Lambda')$. Take an element $g \in \mathrm{P}^0(\Lambda)$. Then its reduction $\bar{g}$ modulo $\tilde{\mathfrak{a}}_1(\Lambda)$ projects to an element of $\mathrm{SU}(\bar{h})$. Thus $g\Lambda_{\frac{1}{2}} = \Lambda_{\frac{1}{2}}$, and thus g is an element of $\mathrm{P}(\Lambda')$.

2) We show $P_1(\Lambda) = P_1(\Lambda')$. The group $P_1(\Lambda)$ is contained in $P_1(\Lambda')$ because the image of Λ' contains the image of Λ. To prove the other inclusion take an element $g = 1 + x \in P_1(\Lambda')$. It acts as the identity in the isotropic space $\Lambda_{\frac{1}{2}}/\Lambda_1$. The only element of $O(1,1)(k_D)$ which coincides with the identity on one isotropic space is the identity itself. Therefore $x\Lambda_0 \subseteq \Lambda_1$ and g has to be an element of $P_1(\Lambda)$.

3) We show $P^0(\Lambda)$ is contained in $P^0(\Lambda')$. The quotient map ϕ from $\tilde{\mathfrak{a}}(\Lambda')/\tilde{\mathfrak{a}}_1(\Lambda)$ to $\tilde{\mathfrak{a}}(\Lambda')/\tilde{\mathfrak{a}}_1(\Lambda')$ is a k_D-algebra map which over the algebraic closure of k_D maps the group $\mathbb{P}^0(\Lambda)$ into $\mathbb{P}(\Lambda')$. Thus $\mathbb{P}^0(\Lambda')$ contains $\phi(\mathbb{P}^0(\Lambda))$ and we obtain

$$\begin{aligned} P^0(\Lambda)/P_1(\Lambda') &= \phi(P^0(\Lambda)/P_1(\Lambda)) \\ &= \phi(\mathbb{P}^0(\Lambda)(k_D)) \\ &= \phi(\mathbb{P}^0(\Lambda))(k_D) \\ &\subseteq \mathbb{P}^0(\Lambda')(k_D) \\ &= P^0(\Lambda')/P_1(\Lambda'). \end{aligned}$$

The 3rd equality arises, because k_D is perfect: use that ϕ is defined over k_D, pass to the algebraic closure and use the Galois action of $\mathrm{Gal}(\bar{k}_D|k_D)$. Now, the desired containment follows from 2).

4) It remains to prove: $P^0(\Lambda')$ is contained in $P^0(\Lambda)$. From 3) we get the following for algebraic groups over k_D:

$$\mathbb{P}(\Lambda) \supseteq \mathbb{U}(\tilde{\mathfrak{a}}(\Lambda')/\tilde{\mathfrak{a}}_1(\Lambda), \bar{\sigma}_h) \overset{\phi\times\bar{k}_D}{\longrightarrow} \mathbb{U}(\tilde{\mathfrak{a}}(\Lambda')/\tilde{\mathfrak{a}}_1(\Lambda'), \bar{\sigma}_h) = \mathbb{P}(\Lambda'),$$

and the derivative of ϕ at the identity is an isomorphism, because $\mathfrak{a}_1(\Lambda') = \mathfrak{a}_1(\Lambda)$. Thus the kernel of $\phi \times \bar{k}_D$ is finite, and therefore $\mathbb{P}^0(\Lambda), \mathbb{P}^0(\Lambda')$ and $\phi(\mathbb{P}^0(\Lambda))$ have the same dimension, i.e. the latter two algebraic groups coincide. This finishes the proof. □

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