(c) $M = \mathbb{R}P^2$ M is the image of the map

$$S^2 \xrightarrow{\pi} \mathbb{R}P^2$$

$$(x,y,z) \longmapsto [x:y:z]$$

(π identifies antipodal points)

U = Möbius strip (open)

V = open disc.

$U \cup V = \mathbb{R}P^2$, $\quad U \underset{\text{homotopic}}{\simeq} S^1$

$$V \sim pt$$
$$U \cap V \sim S^1$$

Mayer-Vietoris sequence:

$$0 \to \underbrace{H^0(\mathbb{R}P^2)}_{\mathbb{R}} \to \mathbb{R} \oplus \mathbb{R} \xrightarrow{\beta^{(0)}} \overset{\mathbb{R}}{\underset{\text{is}}{}} H^0_{dR}(S^1) \xrightarrow{\delta^{(0)}}$$

$$\xrightarrow{\delta^{(0)}} H^1_{dR}(\mathbb{R}P^2) \xrightarrow{\alpha^{(1)}} 0 \oplus \mathbb{R} \xrightarrow{\beta^{(1)}} \overset{\mathbb{R}}{H^1_{dR}(S^1)}$$

$$\xrightarrow{\delta^{(1)}} H^2_{dR}(\mathbb{R}P^2) \longrightarrow 0$$

$\beta^{(1)}$ is the multiplication with 2
$\mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto 2x$. (Why?)

Thus $\delta^{(1)}$ is the 0-map $\Rightarrow \underset{0}{\overset{}{H^2_{dR}(\mathbb{R}P^2)}} $ $= 0$.

Further $\beta^{(0)}$ is surjective $\Rightarrow \alpha^{(1)}$ is injective

$\Rightarrow \quad H^1_{dR}(\mathbb{R}P^2) \cong \operatorname{im} \alpha^{(1)} = \ker(\beta^{(1)}) = 0$

$\Rightarrow \quad H^1_{dR}(\mathbb{R}P^2) = 0$

Thus
$$H^i_{dR}(\mathbb{R}P^2) \cong \begin{cases} \mathbb{R}, & i = 0 \\ 0, & i > 0. \end{cases}$$

(d) Exercise: Compute $H^*_{dR}(\mathbb{R}P^n)$ for $n > 2$.

Chapter IV Integration on manifolds

As indicated in Question 3.6 in Chapter III we have to choose an atlas where all overlaps have positive determinant. This is encoded in the concept of orientation.

Def. 4.1: Let V be a finite dimensional (f.d.) real v.s..
We say that two bases $(v_1, v_2, \dots, v_n)$ and $(w_1, \dots, w_n)$ have the same orientation if the base change matrix from $(\underline{v})$ to $(\underline{w})$ has positive determinant.
This defines an equivalence relation on the set of ordered bases of V.
We denote the equivalence classes by

$[(\underline{v})]$, and the factor set by

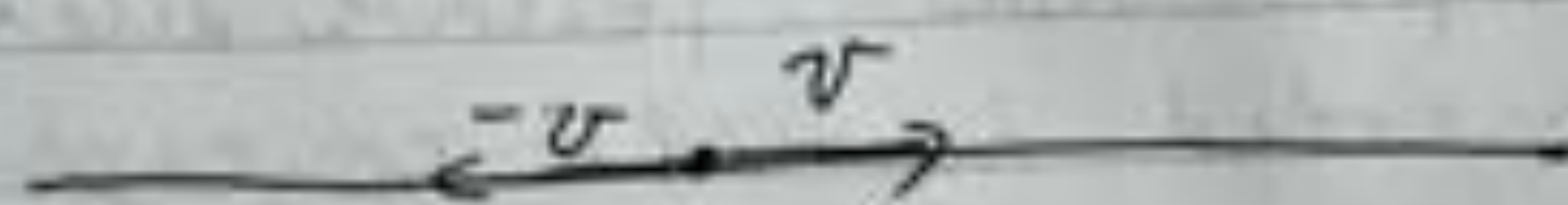
$$\operatorname{or}(V) := \{ [(\underline{v})] \mid (\underline{v}) \text{ ordered basis of } V \}$$

$[(\underline{v})]$ is called an orientation on V.

Example 4.2:

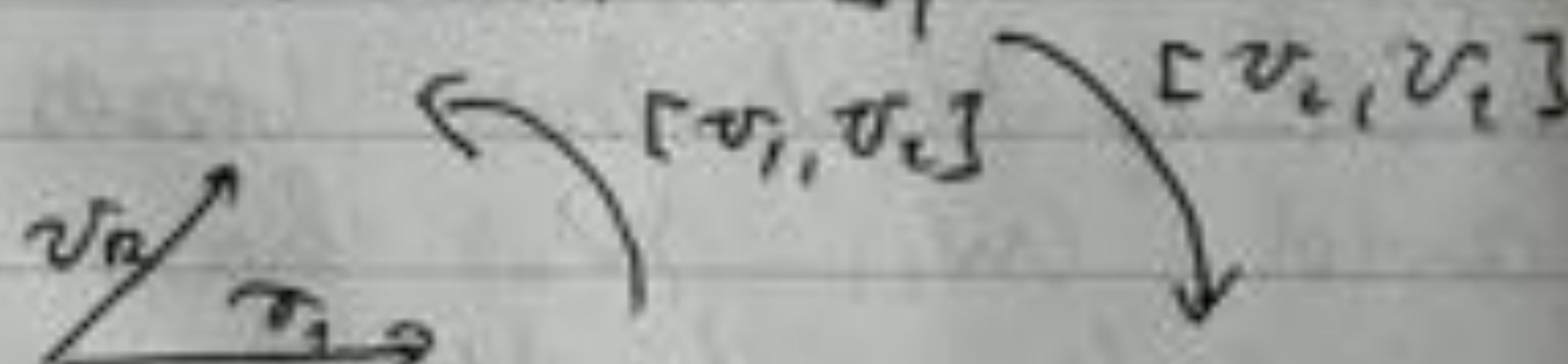
(a) $V = \mathbb{R}v$ 1-dim.

$$or(V) = \{ [v], [-v] \}$$



(b) $V = \mathbb{R}v_1 \oplus \mathbb{R}v_2$ 2-dim

$$or(V) = \{ [(v_1, v_2)], [(v_2, v_1)] \}$$

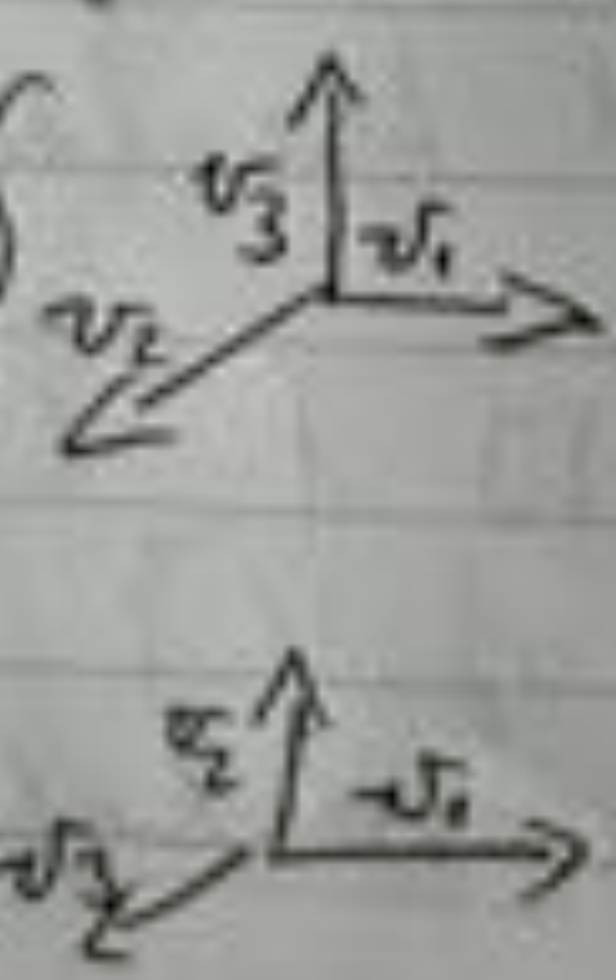


In particular $[(-v_1, v_2)] = [(v_2, v_1)]$

(c) $V = \mathbb{R}v_1 \oplus \mathbb{R}v_2 \oplus \mathbb{R}v_3$

$$or(V) = \{ [(v_1, v_2, v_3)], [(v_1, v_3, v_2 \ldots$$

$[(v_1, v_2, v_3)]$ in $\mathbb{R}^3$



($[(v_1, v_2, v_3)]$ oriented according to left hand rule starting with thumb)

($[(v_1, v_2, v_3)]$ oriented according to the right hand rule)

Let M be a mf.

Def 4.3: A map $\sigma : M \longrightarrow \bigsqcup_{p \in M} or(T_pM)$

is called an orientation on M if $\forall p \in M$ $\exists$ chart (φ, U) around p such that for all $Q \in U$ we have

$$\sigma(Q) = \left[\frac{\partial}{\partial x_1}(Q), \frac{\partial}{\partial x_2}(Q), \dots, \frac{\partial}{\partial x_m}(Q)\right].$$

We denote by $or(M)$ the set of orientations on M.

$$\left(or(M) := \{\sigma : M \longrightarrow \bigsqcup_{p \in M} or(T_pM) \mid \sigma \text{ is an orientation on } M\}\right)$$

Remark 4.4: (a) If M is connected then $|or(M)| \in \{0, 2\}$.

Pf: If $\sigma, \sigma' \in or(M)$ then

$U_= := \{p \in M \mid \sigma(p) = \sigma'(p)\}$ and
$U_{\neq} := \{p \in M \mid \sigma(p) \neq \sigma'(p)\}$ are open and $M = U_{\neq} \dot\cup U_=$.

Thus $M = U_=$ or $M = U_{\neq}$.

Take $p_0 \in M$. Then therefore the map $\operatorname{or}(M) \longrightarrow \operatorname{or}(T_{p_0}M)$

$$\sigma \longmapsto \sigma(p_0)$$

is injective.

$\Rightarrow \quad |\operatorname{or}(M)| \leq 2.$

To show $|\operatorname{or}(M)| \neq 1$: Exercise. $\square$

<u>Prop 4.5:</u> Let M^m be a mf. Then are equivalent:

1° $\operatorname{or}(M) \neq \emptyset$ (We say M is <u>orientable</u>.)

2° M has an atlas where all the transition maps (i.e. the overlaps) have positive determinant.

3° $\exists\, \omega \in \Omega^m(M)$: ω vanishes nowhere, i.e. $\forall p \in M$: $\omega_p \neq 0$.

<u>Proof:</u> 1° $\Leftrightarrow$ 2° Exercise. It is essentially the definition

2° $\Rightarrow$ 3° Exercise: Use partition of unity.

$3^\circ \Rightarrow 1^\circ$ Take $\omega \in \Omega^m(M)$ vanishing nowhere. We define

$$\sigma : M \longrightarrow \bigsqcup_{p \in M} \mathrm{or}(T_pM)$$

via $\sigma(p) := \{(\underline{v}) \mid$

$v_1, \ldots, v_m \in T_pM$ and $\omega(v_1, \ldots, v_m) > 0\}$.
We have to show $\sigma \in \mathrm{Or}(M)$.
Take $p \in M$ and (φ, U) a chart around p.
Then $\omega|_U = a \, dx_1 \wedge \ldots \wedge dx_m$.

with $a \in C^\infty(U)$ nowhere vanishing. We take U to be connected. Then $a(U) \subseteq]0, \infty[$ or $a(U) \subseteq]-\infty, 0[$.
<u>Case</u> $a(U) \subseteq]0, \infty[$: Then

$$\sigma(Q) = [(\frac{\partial}{\partial x_1}(Q), \ldots, \frac{\partial}{\partial x_m}(Q))]$$

for all $Q \in U$.

<u>Other case:</u> $\sigma(Q) = [-\frac{\partial}{\partial x_1}(Q), \frac{\partial}{\partial x_2}(Q), \ldots, \frac{\partial}{\partial x_m}(Q)]$

for all $Q \in U$. So here take the chart $((-x_1, x_2, \ldots, x_m), U)$. $\square$

Example 4.6.: (a) $\mathbb{R}^n$ is orientable

because $\omega = dx_1 \wedge \dots \wedge dx_n \in \Omega^n(\mathbb{R}^n)$ is nowhere vanishing:
at $p \in \mathbb{R}^n$ we have $\omega_p\left(\frac{\partial}{\partial x_1}(p), \dots, \frac{\partial}{\partial x_n}(p)\right) = 1$.

(b) S^n is orientable, because S^n is the boundary of an orientable manifold:
$S^n = \partial \overline{B_1(0)}$.
See the following proposition.

(c) Let N be a submanifold of M of the same dimension m and let σ be an orientation on M. Then $\sigma|_N$ is an orientation on N.
(Caution: M and N could have boundary)
(Exercise!)

(d) The Möbius strip M is not orientable
Proof: Assume M is orientable. Then $\exists\, \omega \in \Omega^2(M)$

st ω is nowhere vanishing
Now take $E_i : \mathbb{R} \longrightarrow TM$

from Example 1.29 (b).

Then $(E_1(\theta), E_2(\theta))$ is an ordered basis of $T_{P(\theta,0)}M$ for all $\theta \in \mathbb{R}$.

Then $f(\theta) := \omega_{P(\theta,0)}(E_1(\theta), E_2(\theta))$

defines a nonvanishing C^∞ map
$f : \mathbb{R} \longrightarrow \mathbb{R}$.
Now $f(0) = \omega_{(1,0)}(E_1(0), E_2(0))$

and $f(2\pi) = \omega_{(1,0)}(-E_1(0), E_2(0))$

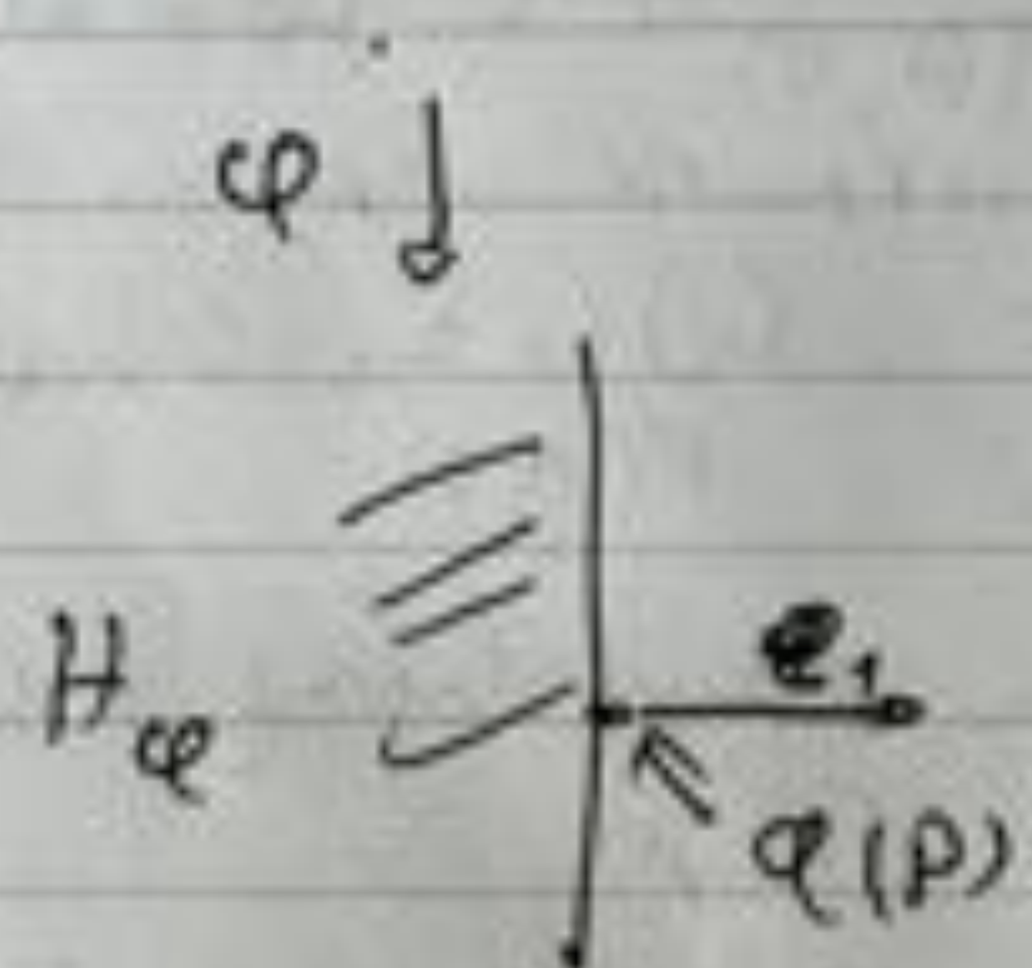
$$= -f(0).$$

Thus f has a positive and a negative value. $\lightning$ to the intermediate value theorem.

<u>Proposition 4.7:</u> Let $M^{m},\ m \geq 2$ be a mf and $\partial M \neq \emptyset$. Suppose $or(M) \neq \emptyset$. Then $or(\partial M) \neq \emptyset$.

Proof: Take $\sigma \in or(M)$.
For $p \in \partial M$ and a chart $(\varphi = (x_1, \dots, x_m), U)$ preserving σ s.t. $\frac{\partial}{\partial x_1}(p)$ is pointing outwards

The half space H_φ



We define $\partial\sigma(p) := \{ (\underline{w}) \mid (\underline{w})$ ordered basis of $T_p \partial M$ such that $[\frac{\partial}{\partial x_1}, w_1, \dots, w_{m-1}] = \sigma(p) \}$

Exercise $\partial\sigma : \partial M \longrightarrow \coprod_{p \in \partial M} Or(T_p \partial M)$ is an orientation on ∂M. □

Example 4.8.

$\partial \bar{B}_1(\underline{0}) = S^2$

$[(w_1, w_2)]$ comes from the right left hand orientation of $\mathbb{R}^3$.

Remark / Def. 4.9: We also need the notion of orientation of a 0-dim manifold.

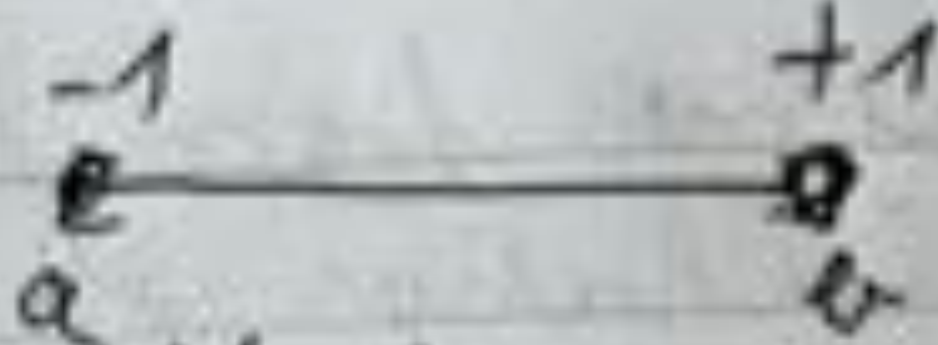
Let M be a zero dimensional mf. then M is a countable set.
An orientation on M is a map
$$\sigma : M \longrightarrow \{\pm 1\}.$$

Prop 4.7 still holds for M^1 with $\partial M^1 \neq \emptyset$. ($\dim \partial M^1 = 0$)
Let $\sigma : M \longrightarrow \bigsqcup_p or(T_p M)$ be an orientation on M.
$p \in \partial M$. Let $(\varphi = (x), U)$ a chart with $p \in U \cap A$. $\left[\frac{\partial}{\partial x}(p)\right] = \sigma(p)$.

We choose $\partial\sigma(p) \in \{\pm 1\}$ such that $\partial\sigma(p) \frac{\partial}{\partial x}(p)$ is pointing outwards.

Ex: $M := [a, b] \subseteq \mathbb{R}$
$$\sigma : M \longrightarrow \bigsqcup_{p \in [a,b]} \mathbb{R} e_1$$
$$\sigma(p) := [(p, e_1)] = \left[\frac{\partial}{\partial x}(p)\right]$$

$$\partial M = \{a, b\}$$



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Example 4.10 a) $M = \overline{B}_1(\underline{0}) \subseteq \mathbb{R}^2$

σ given by $dx_1 \wedge dx_2$.

$\partial\sigma$ is the counter clockwise orientation on $S^1 = \partial M$.

b)

$M = \overline{B}_1(\underline{0}) \subseteq \mathbb{R}^3$
$\partial M = S^2$

σ given by $dx_1 \wedge dx_2 \wedge dx_3$

$\partial\sigma$ counter clock-wise w.r.t the outwards pointing normal vector (right hand rule)

Def 4.11: (integration)

(a) X top space. $C_c(X, \mathbb{R}) := \{ f: X \to \mathbb{R} \mid f$ continuous with compact support$\}$

($\mathrm{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}$.)

(b) Let M^m $(m \geq 1)$ be a mf with orientation σ. An atlas $(\varphi_i, U_i)_{i \in I}$ is called to have orientation σ, if every chart (φ_i, U_i) has orientation $\sigma|U_i$.

A chart $(\overset{(y_1, \dots, y_m)}{\psi}, V)$ is said to have orientation $\sigma|_V$ if

$\sigma(Q) = [\frac{\partial}{\partial y_1}(Q), \dots, \frac{\partial}{\partial y_m}(Q)]$ $\forall_{Q \in V}$.

(c) Let M^m be a mf. and $k \in \mathbb{N}_0$

$\Omega_c^k(M) = \{ \omega \in \Omega^k(M) \mid \mathrm{supp}(\omega)$ is compact$\}$

(d) Let M^m, $m \geq 1$, be a mf and (ψ, V) be a chart. We define

$\overset{(\underline{y})}{\overset{\shortparallel}{\psi}}$

$I_{(\psi, V)} : \Omega_c^m(V) \longrightarrow \mathbb{R}$

via $I_\psi(\omega) = \int_{\psi(V)} \overbrace{a(\psi^{-1}(y))}^{a(\psi^{-1}(y_1,\dots,y_m))}\, dy_1\, dy_2 \dots dy_m$

for $\omega = a\, dy_1 \wedge dy_2 \wedge \dots \wedge dy_m \in \Omega^m_c(V)$

(e) Let (M^m, σ) be an oriented mf with $m \geq 1$. We define

$$I_{(M,\sigma)} : \Omega^m_c(M) \longrightarrow \mathbb{R}$$

as follows.
Let $(U_i, \varphi_i)_{i \in I}$ be a locally finite atlas with orientation σ and $(\lambda_i)_{i \in I}$ be a partition of unity adapted to $(U_i)_{i \in I}$.

We put for $\omega \in \Omega^m_c(M)$:

$$I_{(M,\sigma)}(\omega) := \sum_{i \in I} I_{\varphi_i}(\lambda_i \omega)$$

(f) Let (M^0, σ) be an oriented 0-dimensional mf...

We define $I_{(M,\sigma)} : \Omega_c^0(M) \longrightarrow \mathbb{R}$ as follows: $I_{(M,\sigma)}(a) = \sum_{p\in M} a(p)\cdot\sigma(p)$, $a \in C_c(M, \mathbb{R})$.

<u>Notation / Terminology 4.12:</u>

We call $I_{(M,\sigma)}(\omega)$ the <u>integral of ω over M w.r.t. σ</u>, and we write $\int_{M,\sigma} \omega$.

<u>Prop. 4.13:</u> Let (M^m, σ) be an oriented mf. with $m \geq 1$ and $\omega \in \Omega_c^m(M)$. Then $I_{(M,\sigma)}(\omega)$ is well-defined.

<u>Proof:</u> <u>Step 1:</u> Let (φ, U) and $(\psi, V = U)$ be two charts with $D(\varphi\circ\psi^{-1})(\underline{y})$ having positive determinant for all $\underline{y} \in \psi(U)$. Suppose $\omega \in \Omega_c^m(U)$.

Then $I_\varphi(\omega) = I_\psi(\omega)$.

$$\omega = a\, dx_1 \wedge \dots \wedge dx_m$$

$$= a \underbrace{\det(D(\varphi \circ \psi^{-1}) \circ \psi)}_{>0} dy_1 \wedge \dots \wedge dy_m$$

The rest follows from the transformation formula.

Step 2: Independence of atlas and partition of unity. (Exercise!) ∎

Examples 4.14: (a) $M = \mathbb{Z} \subseteq \mathbb{R}$
$\dim M = 0$ and M has countably many connected components.

$$C_c(M, \mathbb{R}) = \{ f : \mathbb{Z} \to \mathbb{R} \mid \exists\, q \in \mathbb{R}^{>0} \underset{\substack{z \in \mathbb{Z} \\ |z| \geq q}}{\forall} : f(z) = 0 \}$$

Take the orientation:

$\sigma : M \longrightarrow \{\pm 1\}$
$\sigma(z) = (-1)^z.$
For $f \in C_c(M, \mathbb{R}) = \Omega^0_c(M)$:

$$\int_{M,\sigma} f = \sum_{z \in \mathbb{Z}} (-1)^z f(z).$$

(b) $M = [a, b] \subseteq \mathbb{R}$
σ given by $dx|_M \in \Omega^1(M)$
(dx is nowhere vanishing)
<u>Note</u> that we do <u>not</u> have a global chart for M in a strict sense!

$U := [a, c[\,, \; V =]d, b]$

with $a < d < c < b$.
$(\varphi, U) = (id_U, U)$ $\quad \varphi = (s)$
$(\psi, V) = (id_V, V)$ $\quad \psi = (t)$

$\sigma(b) = [\frac{\partial}{\partial x}(b)] = [1 \cdot \frac{\partial}{\partial t}(b)]$

and $\frac{\partial}{\partial t}(b)$ is pointing outwards

$$\sigma(a) = \left[\frac{\partial}{\partial x}(a)\right] = \left[\frac{\partial}{\partial s}(a)\right]$$
$$= \left[(-1)\left(-\frac{\partial}{\partial s}(a)\right)\right]$$

and $-\frac{\partial}{\partial s}(a)$ is pointing outwards. Thus

$$\partial\sigma(a) = -1 \text{ and } \partial\sigma(b) = 1.$$

Now we compute some integrals.

(b1) $\omega \in \Omega^1(M)$

Then $\omega = a|_M \, dx|_M$ for some $a \in C^\infty(\mathbb{R})$.

(Note $C^\infty(M, \mathbb{R}) = \{ b|_M \mid b \in C^\infty(\mathbb{R}) \}$
$= \{ b|_M \mid b \in C_c^\infty(\mathbb{R}) \}$

Take a partition of unity (λ_U, λ_V) adapted to (U, V).

$$I_{(M,\sigma)}(\omega) = I_{(\varphi,U)}(\lambda_U \omega) + I_{(\psi,V)}(\lambda_V \omega)$$
$$= \int_{[a,c[} \lambda_U(s)\, a(s)\, ds + \int_{]d,b]} \lambda_V(\ldots$$

$$= \int_a^a \lambda_u(r)a(r)\,dr + \int_a^a \lambda_v(r)a(r)\,dr$$

Riemann integrals

$$= \int_a^b a(r)\,dr.$$

(b2) $\omega \in \Omega^1(M)$ exact.
$\omega = df$ for some $f \in C^\infty(M)$
$\quad\;\; \| \;\; f'(x)\,dx$ (exercise!)

$$I_{(M,\sigma)}(\omega) \underset{(b1)}{=} \int_a^b f'(r)\,dr \underset{FTC}{=} f(b) - f(a)$$

$$= I_{(\partial M, \partial\sigma)}(f|_{\partial M}).$$

This is the baby version of Stokes' theorem.

(c) $M = S^2$, orientation $= \sigma_M = \partial\sigma_{\bar{B}_1(\underline{0})}$
(outwards pointing normal vector.)

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(d)

$$\int_{S^2} z\,dx\wedge dy = \int_{\substack{U_+ \\ \shortparallel \\ S^2\cap\{z>0\}}} z\,dx\wedge dy + \int_{\substack{U_- \\ \shortparallel \\ S^2\cap\{z<0\}}} z\,dx\wedge\dots$$

$$= \int_{B_1(0)^{(2)}} z_+(x,y)\,dx\,dy + \underset{\substack{\uparrow \\ \text{by the orientation}}}{(-1)} \int_{B_1(0)^{(2)}} z_-(x,y)\,dx\dots$$

(The orientation is given by the form $z\,dx\wedge dy + x\,dy\wedge dz + y\,dz\wedge d\dots$

$$= 2\int_{B_1(0)^{(2)}} \sqrt{1-x^2-y^2}\,dx\,dy$$

$$= 2\int_0^1\int_0^{2\pi} \sqrt{1-r^2}\,r\,d\theta\,dr$$

$$= 2\left(-\frac{1}{2}\right)\cdot 2\pi\,\frac{2}{3}(1-r^2)^{\frac{3}{2}}\Big|_0^1 = \frac{4}{3}\pi.$$

(d) $M = \overline{B_1(0)}^{(3)} \subseteq \mathbb{R}^3$. $\sigma_M = \sigma_{\mathbb{R}^3}\big|_M$

$$\int_M dz \wedge dx \wedge dy = \int_{\overline{B}_1(0)^{(3)}} dx\, dy\, dz$$

$$= \int_0^1 \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 \cos\theta_2 \; d\theta_2\, d\theta_1\, dr$$

$$= \left(\frac{1}{3} r^3 \Big|_0^1\right) \cdot 2\pi \cdot \sin\theta_2 \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}.$$

$$= \frac{4}{3}\pi.$$

Theorem 4.15 (Stoke's Theorem)

Let M^m be a oriented manifold and $\omega \in \Omega_c^{m-1}(M)$. Then

$$\int_M d\omega = \int_{\partial M} \omega|_{\partial M}$$

$(m \geq 1)$.

See Example 4.14. (b) and (c) with (d).

Def. 4.16: A manifold is called closed if it is compact without boundary. (M, M compact and $\partial M = \emptyset$)

Corollary 4.16: Let M^m be a closed oriented manifold, $m \geq 1$, and $\omega \in \Omega^{m-1}(M)$. Then $\int_M d\omega = 0$.

Proof: $$\int_M d\omega \overset{\text{Stoke}}{=} \int_{\partial M} \omega \underset{\partial M = \emptyset}{=} 0. \quad \square$$

Proof of Theorem 4.15.:

Step 1: (Reduction to the case of a global chart.)

$\omega \in \Omega_c^{m-1}(M)$ Let $(\lambda_i)_{i\in I}$

be a partition of unity subordinate to an atlas $(\varphi_i, \underline{U_i})_{i\in I}$ adapted to σ_M such that $\overline{U_i}$ is compact for all $i \in I$.

Then $\lambda_i \omega \in \Omega_c^{m-1}(U_i)$

and $\partial U_i = U_i \cap \partial M$.

and only for finitely many $i \in I$ we have $\operatorname{supp} \omega \cap \operatorname{supp} \lambda_i \neq \phi$, say for $i \in I_0$.

Suppose we have Stokes's theorem for U_i.

Then $\int_M d\omega \underset{\uparrow}{=} \sum_{i\in I_0} \int_M d(\lambda_i \omega)$

$d\omega = d(\Sigma \lambda_i \omega) = \Sigma\, d(\lambda_i \omega)$

$$= \sum_{i\in I_0} \int_{U_i} d(\lambda_i \omega) \overset{\text{Stoke for } U_i}{=} \sum_{i\in I_0} \int_{\partial U_i} (\lambda_i\omega)|_{\partial U_i}$$

$$\underset{\partial U_i = (\partial M)\cap U_i}{=} \sum_{i\in I_0} \int_{(\partial M)\cap U_i} (\lambda_i \omega)|_{\partial U_i}$$

$$\underset{\left(\operatorname{supp}(\lambda_i\omega)|_{\partial U_i}\right) \subseteq \partial U_i \text{ and is compact}}{=} \sum_{i\in I_0} \int_{\partial M} (\lambda_i\omega)|_{\partial M} = \int_{\partial M} \left(\Sigma_{i\in I_0} \lambda_i \omega\right)|_{\partial M}$$

$$= \int_{\partial M} \omega|_{\partial M}.$$

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<u>Step 2:</u> $M = U \subseteq \mathbb{R}^m$ open.

$$\omega = \sum_{\substack{|J| = m-1 \\ J \subseteq \{1,\dots,m\}}} a_J \, dx^J \in \Omega_c^{m-1}(U)$$

$$\Rightarrow d\omega = \sum_J da_J \wedge dx^J$$

$$= \sum_J \frac{\partial a_J}{\partial x_{i_J}} dx_{i_J} \wedge dx^J$$

(i_J is the elt of $\{1,\dots,m\}$ not in J)

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$$\int_{\partial U} \omega = 0 \text{ because } \partial U$$

(the mf. boundary) is empty.

$$\int_U d\omega = \sum_j \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\partial a_j}{\partial x_{i_j}} dx_1 dx_2 \cdots dx_m \cdot (-1)^{i_j+1}$$

$$= \sum_j \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \cdots dx_{i_j}}_{\text{by FTC} = 0} dx_1 \cdots \widehat{dx_{i_j}} \cdots dx_m$$

$= 0$

<u>Step 3</u>: $(m \geq 2)$ $M = U \subseteq H = \{x_1 \leq 0\} \subseteq \mathbb{R}^m$

U open in H s.t. $\partial U = \partial H \cap U$.
$= U \cap \{x_1 = 0\}$.

$$\Rightarrow \int_U d\omega = \sum_j \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{0} \frac{\partial a_j}{\partial x_{i_j}} dx_1 \cdots dx_m \, (-1)^{i_j+1}$$

$$\underset{\uparrow}{=} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{0} \frac{\partial a_0}{\partial x_1} dx_1 \cdots dx_m$$

summand for $J \neq \{2, \dots, m\}$ is zero

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} a_0(0, x_2, \dots, x_m) dx_2 \cdots dx_m$$

$$\underset{\uparrow}{=} \int_{\partial U} a_0 \, dx_2 \wedge \dots \wedge dx_m$$

$dx_2 \wedge \dots \wedge dx_m$ gives the orientation $\partial \sigma_M$.

$$\underset{\uparrow}{=} \int_{\partial U} \omega|_{\partial U}$$

$\left(a_J dx^J|_{\partial U} = 0 \text{ for } J \neq \{2, \dots, m\} \right)$

<u>Step 4</u>: $(m = 1)$ $\quad M = U \subseteq H$ $\begin{cases} \{x_1 \leq 0\} \\ \{x_1 \geq 0\} \end{cases} \subseteq \mathbb{R}$.

U open in H s.t. $\partial U = \{0\}$.
Similar to Step 3. (Exercise.) $\square$

Example 4.17

(a) Fundamental Theorem for line integrals.

Let C be a smooth curve in $\mathbb{R}^3$ parametrized by $r(t) = (x(t), y(t), z(t))$, $t \in [a, b]$ such that $r'(t) \neq 0$ for all t. Then for $f \in C^\infty(\mathbb{R}^3)$ we have

$$\int_C df = \int_{\partial C} f = f(r(b)) - f(r(a))$$

$$\left(df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right)$$

(b) Green's theorem

Let D be a compact ~~set~~ submf of $\mathbb{R}^2$ of dimension 2 and $P, Q \in C^\infty(\mathbb{R}^2)$. Then

$$\int_{\partial D} P\,dx + Q\,dy = \int_D \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy$$

$$= \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

(Here $\sigma_D = \sigma_{\mathbb{R}^2}|_D$)

Important forms to integrate are volume forms. Ld.
Recall: A k-tensor on a v.o. V is a k-linear map $f: \underbrace{V \times \dots \times V}_{k} \to \mathbb{R}$, $k \geq 1$.

Def 4.18: A k-tensor $f: V \times \dots \times V \to \mathbb{R}$ is called symmetric if

$$f(v_1, \dots, v_k) = f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \; \forall \sigma \in \mathfrak{S}_k$$

$\mathrm{Sym}^k(V) = \{ f: V \times \dots \times V \to \mathbb{R} \mid f$ a symmetric k-tensor on $V.\}$

Def 4.19: Let $k \in \mathbb{N}$, $\sigma \in \mathfrak{S}_k$, V be a f.d. real v.s.
σ induces an isomorphism

$$\sigma: V^{\otimes k} \longrightarrow V^{\otimes k}$$

given by $\sigma(v_1 \otimes \dots \otimes v_k) = v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)}$

The $\mathbb{R}$-vector space

$$S^k(V) := V^{\otimes k} \Big/ \{A - \sigma(A) \mid A \in V^{\otimes k}, \sigma \in \mathfrak{S}_k\}$$

is called the <u>kth symmetric power of V</u>

<u>Prop 4.20</u> Let $k \in \mathbb{N}$ and V be a f.d. real vector space. Then

$$S^k(V^*) \xrightarrow[(v_1,\dots,v_k)]{\overset{\Phi}{\sim}} \mathrm{Sym}^k(V) \text{ via}$$

$$\Phi([\omega_1 \otimes \dots \otimes \omega_k]) := \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \omega_1(v_{\sigma(1)}) \cdots \omega_k(v_{\sigma(k)})$$

<u>Proof:</u> Φ is well-defined (Exercise)

We construct $\Psi : \mathrm{Sym}^k(V) \to S^k(V^*)$ such that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are identities.

$$\begin{tikzcd}
\mathrm{Mult}^k(V) \arrow[r, "\sim", "\alpha"'] & T^k(V^*) = V^{*\otimes k} \arrow[d] \\
\mathrm{Sym}^k(V) \arrow[u, phantom, "\cup" rotate=90] \arrow[r, "\Psi"] & S^k(V^*)
\end{tikzcd}$$

Take a basis $e_1, \dots, e_m$ of V with dual $e_1^*, \dots, e_m^*$.

$$\Psi(s) = \Big[\sum_{(i_1,\dots,i_k)} s(e_{i_1},\dots,e_{i_k})\, e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*\Big]$$

$$= \sum_{(i_1,\dots,i_k)} s(e_{i_1},\dots,e_{i_k}) \big[e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*\big]$$

So $\Psi(\Phi([e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*])) =$

$$= \frac{1}{k!} \sum_{\varsigma \in \mathfrak{S}_k} \big[e_{i_{\varsigma(1)}}^* \otimes \cdots \otimes e_{i_{\varsigma(k)}}^*\big]$$

$$= \big[e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*\big].$$

$\Phi(\Psi(s))(e_{j_1}, \dots, e_{j_k})$

$$= \sum_{(i_1,\dots,i_k)} s(e_{i_1},\dots,e_{i_k}) \frac{1}{k!} \sum_{\varsigma \in \mathfrak{S}_k} \delta_{(i_1,\dots,i_k)(j_{\varsigma(1)},\dots,j_{\varsigma(k)})}$$

$$= \frac{1}{k!} \sum_{\varsigma \in \mathfrak{S}_k} \sum_{(i_1,\dots,i_k)} \delta_{(i_1,\dots,i_k)(j_{\varsigma(1)},\dots,j_{\varsigma(k)})}\, s(e_{j_1},\dots,e_{j_k})$$

$$= s(e_{j_1},\dots,e_{j_k}) \qquad \square$$

$$\cancel{\frac{1}{k!}\sum_{\rho\in S_k} a_1!\cdots a_k!\,\rho(e_{j_1},\dots,e_{j_k})}$$

~~repetitions in $(j_1,\dots,j_k)$~~

Remark 4.21: (1) $S^k(V)$ can be identified with

$$\{A\in T^k(V) \mid \rho(A)=A \ \forall \rho\in S_k\}$$

given by Φ and $\mathrm{Mult}^k(V^*)\simeq T^k(V)$

Ex: $v_1 v_2 \triangleq \frac{1}{2} v_1\otimes v_2 + \frac{1}{2} v_2\otimes v_1$

(2) A basis on $S^k(V)$ is given by

$$(e_{i_1} e_{i_2} \cdots e_{i_k})_{1\le i_1\le i_2\le\dots\le i_k}$$

$$\parallel$$

$$[e_{i_1}\otimes\cdots\otimes e_{i_k}]$$

(3) k-th symmetric power of the cotangent bundle

$$S^k(T^*M)\longrightarrow M.$$

Def 4.22: (a) A metric tensor on a f.d. real v.s. V is an element $s \in S^2(V^*)$ which is non-degenerate, i.e.

$s(v, -)$ is ~~[illegible]~~ non-zero for all $v \neq 0$

(b) Let M be a mf. metric tensor field g (or just metric) on M is a smooth section of

$S^2(T^*M) \longrightarrow M$

s.t. $g(p)$ is a metric tensor $\forall p \in M$.

Example 4.23: (a) $dx_1 \otimes dx_1 + \ldots + dx_n \otimes dx_n$

is a metric on $\mathbb{R}^n$.

(b) Let $A = (a_{ij})$ be a symmetric metric. Then

$$\sum_{(i,j)} a_{ij} \, dx_i \otimes dx_j \text{ is a metric}$$

on $\mathbb{R}^n$ iff $\det A \neq 0$.

(c) Let N be a submf of M and

g be a metric on M, Then $g|_N$ is a metric on N, if, for all

$p \in N$, $g_p\big|_{T_pN \times T_pN}$ is non-degenerate

<u>Def 4.24:</u> Let M be a mf and g be a metric on M. g is called a <u>Riemannian metric</u> if $\forall p$: g_p is positive definite.
A manifold equipped with a Riemannian metric is called Riemannian manifold.

Let (M^m, g) be a Riemannian manifold with orientation σ.

The m-form

$\omega_g \in \Omega^m(M)$ given by

$\cancel{\omega_{g,p}}$ $\omega_{g|p} = \sqrt{|g|}\, dx_1 \wedge \dots \wedge dx_m$

in local oriented coordinates

with $|g| = \left|\det\left(\left(g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\right)_{ij}\right)\right|$
is called the <u>volume form</u> <u>for g</u>.

$$vol_g(M) := \int_M \omega_g$$

is called the <u>volume of M</u>, for a compact oriented Riemannian manifold.

<u>Example 4.25</u>:

If N is a submf. of M and g is a Riemannian metric on M then $g|_N := (g_p|_{T_p^* N})_{p \in N}$ is a Riemannian metric on N.

<u>Ex</u>: $S^n \subseteq \mathbb{R}^{n+1}$, $n \geq 1$.
$g = dx_1 \otimes dx_1 + \ldots + dx_n \otimes dx_n$
is a Riemannian metric on $\mathbb{R}^{n+1}$
What are the volume forms for g and $g|_{S^n}$?

<u>For g:</u> $\sqrt{|g|}\, dx_1 \wedge \dots \wedge dx_{n+1}$
$$\shortparallel$$
$$dx_1 \wedge \dots \wedge dx_{n+1}$$

because $\left(g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\right)_{ij} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$.

<u>For $g|_{S^n}$:</u> For $x_{n+1}(p) \gtrless 0$

(x) $\left(\frac{\partial}{\partial x_1} - \frac{x_1}{x_{n+1}} \frac{\partial}{\partial x_{n+1}}, \dots, \frac{\partial}{\partial x_n} - \frac{x_n}{x_{n+1}} \frac{\partial}{\partial x_{n+1}}\right)$

is an <u>oriented frame</u>. Let's denote it $(X_1, \dots, X_n)$.

$$(g(X_i, X_j))_{ij} = I_n + \frac{1}{x_{n+1}^2} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot (x_1, \dots, x_n)$$

has determinant $\frac{1}{x_{n+1}^2}$.

and $dx_1|_{TS^n}, \dots, dx_n|_{TS^n}$ is dual to $X_1, \dots, X_n$.

$\Rightarrow \omega_{g|_{S^n}} = \frac{1}{x_{n+1}} dx_1|_{TS^n} \wedge \dots \wedge dx_n|_{TS^n}$ on $x_{n+1} \gtrless 0$.

In general

$$\omega_{g|_{S^n}} = \frac{1}{x_i} dx_{i+1} \wedge \dots \wedge dx_{i-1}\Big|_{TS^n}$$

on $\{x_i \neq 0\} \cap S^n$.

We compute $vol_{g|_{S^n}}(S^n)$.

$$vol_{g_{S^n}}(S^n) = \int_{S^n} \omega_{g|_{S^n}} \overset{\uparrow}{=} 2 \int_{B_1(\underline{0})^{(n)}} \frac{1}{\sqrt{1-x_1^2-\dots+x_n^2}} dx_1 \dots dx_n.$$

The frame(x) comes from a chart

We consider n-dim. spherical coo.
$(r, \theta_1, \theta_2, \dots, \theta_{n-1})$ for $n \geq 2$.
$\theta_1 \in [0, 2\pi]$, $\theta_2, \dots, \theta_{n-1} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

$x_1 = r \cos(\theta_{n-1}) \cdot \cos(\theta_{n-2}) \cdot \dots \cdot \cos(\theta_2) \cos(\theta_1)$
$x_2 = $ — " — $\sin(\theta_1)$
$\vdots$
$x_{n-2} = r \cdot \cos(\theta_{n-1}) \sin(\theta_{n-2})$
$x_{n-1} = r \sin(\theta_{n-1})$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = J\left(\begin{pmatrix} r \\ \theta_1 \\ \vdots \\ \theta_{n-1} \end{pmatrix}\right)$$

$$|\det(J)| = r^{n-1} \cos^{n-2}\theta_{n-1} \cdots \cos^2\theta_3 \cos\theta_2$$

$vol_g(S^n) = ?$ We need some preparations

① $k \geq 0$: $I_k := \int_0^{\frac{\pi}{2}} \sin^k\theta \, d\theta \underset{\substack{\uparrow \\ \xi = \frac{\pi}{2} - \theta}}{=} \int_0^{\frac{\pi}{2}} \cos^k\theta \, d\theta$

② $I_0 = \frac{\pi}{2}$, $I_1 = 1$, $I_2 \underset{\substack{\uparrow \\ \sin^2\theta + \cos^2\theta = 1}}{=} \frac{\pi}{4}$.

$k \geq 3$ $I_k = \frac{k-1}{k} I_{k-2}$

$$= \begin{cases} \frac{(k-1)!!}{k!!} I_0, & 2 \mid k \\ \frac{(k-1)!!}{k!!} I_1, & 2 \nmid k \end{cases}$$

also valid for $k = 1, 2$.

③ $k \geq 0$, $J_k := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^k\theta \, d\theta = 2 I_k$

④ $k \geq 0$: $H_k = \int_0^1 \frac{r^k}{\sqrt{1-r^2}} \, dr \underset{\substack{\uparrow \\ r = \sin\theta}}{=} \int_0^{\frac{\pi}{2}} \sin^k\theta \, d\theta = I_k$

$$\mathrm{vol}_g(S^1) = 2 \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx = 4 H_0 = 4 I_0 = 2\pi$$

$$\mathrm{vol}_g(S^2) = 2\, H_1\, 2\pi = 4\pi I_1 = 4\pi$$

$$\mathrm{vol}_g(S^3) = 2\, H_2\, 2\pi\, J_1 = 4\pi I_2 \cdot 2 I_1$$

$$= 8\pi \cdot \frac{\pi}{4} = 2\pi^2.$$

$n \geq 4$ $\quad \mathrm{vol}_g(S^n) = 2\, H_{n-1}\, 2\, J_0\, J_1 \cdots J_{n-2}$

$$= 2^{n+1} I_0 \cdots I_{n-1}$$

$$= 2^{n+1} \left(\frac{\pi}{2}\right)^{\lfloor \frac{n+1}{2} \rfloor} \frac{1}{(n-1)!!}$$

also valid for $n = 1, 2, 3$.