The following theorem will be proven later.

<u>Theorem 2.16:</u> Let $M$ and $N$ be $C^s$-manifolds, $1 \le s \le \infty$, and $0 \le r < s$. Then $C^s(M,N)$ is dense in $C^r(M,N)$.

Our goal is

<u>Theorem 2.17:</u> Let $(\alpha_r, M)$ be a $C^r$-manifold $0 \le r < \infty$. Let $r < s \le \infty$.
Then $\exists$ a $C^s$-differential structure $\beta$ such that $\beta \subseteq \alpha_r$.
Two $C^s$-differential structures $\beta, \gamma \subseteq \alpha_r$ ~~differ by a~~ are $C^s$-diffeomorphic, i.e. $\mathrm{Diff}^s((M,\beta),(M,\gamma))$ is non-empty.

We will give the proofs for 2.16 and 2.17 only for mf. without boundary. The more general case can be found in § II 2.3..

From now until the end of Chapter II all manifolds have no boundary.

<u>Remark 2.18:</u> We cannot expect $\beta$ to unique. It is just unique up to $C^s$-diffeomorphis, ~~because~~.

<u>Example:</u> $M = \mathbb{R}$, $\mathcal{A}$ given by $(id, \mathbb{R})$

Take $f : M \longrightarrow M$ defined via

$$f(x) = \begin{cases} (x+\frac{1}{2})^2 - \frac{1}{4}, & x \geq 0 \\ \frac{1}{4} - (-x+\frac{1}{2})^2, & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x+1, & x \geq 0 \\ -2x+1, & x < 0 \end{cases}$$

$$f''(x) = \begin{cases} 2, & x > 0 \\ -2, & x < 0 \end{cases}$$

$$f''(0)_+ = 2 \neq -2 = f''(0)_-$$

$$\Rightarrow f \in C^1(\mathbb{R},\mathbb{R}) \setminus C^2(\mathbb{R},\mathbb{R})$$

(all w.r.t. $(id_{\mathbb{R}}, \mathbb{R})$)

Let $\beta$ be the $C^\infty$-differential structure given by $(id_{\mathbb{R}}, \mathbb{R})$ and put

$$\gamma := \{(\varphi \circ f, f^{-1}(U)) \mid (\varphi, U) \in \beta\}$$

Then $\gamma$ is a $C^\infty$-differential structure on $\mathbb{R}$, but $\cancel{\beta \cap \gamma = \emptyset}$ $\beta \neq \gamma$, because the overlap of $(id_{\mathbb{R}}, \mathbb{R})$ with $(f, \mathbb{R})$ is not $C^\infty$.

Further $f \in \mathrm{Diff}^\infty((\mathbb{R},\gamma),(\mathbb{R},\beta))$ as $\varphi \circ f \circ \psi^{-1} = id_{\mathbb{R}}$ for $\psi = f \in \gamma$ and $\varphi = id \in \beta$.

<u>Lemma 2.19</u> (local modification of $C^r$-morphs)

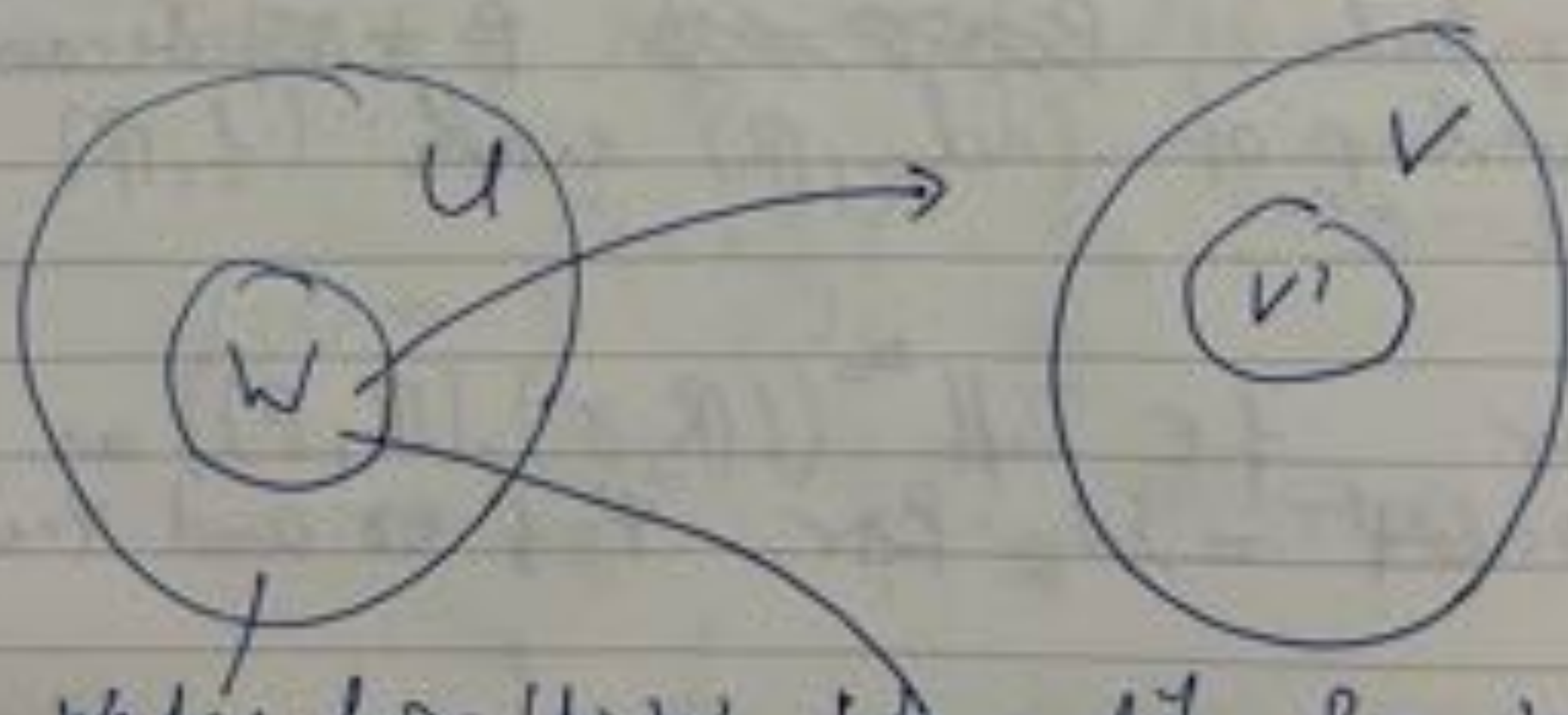
Let $U$ be a $C^r$-mf., $W \subseteq U$ open, $f \in C^r(U, \underset{\text{in } \mathbb{R}^n}{V})$ s.t. $V' := f(W)$ is open in $V$.

Consider the map $T: C^r(W, V') \longrightarrow Map(U,V)$

$$T(g_0) := \begin{cases} g_0, \text{ on } W \\ f, \text{ on } U \setminus W \end{cases}$$

Then $\exists$ $\mathcal{N} \subseteq C^r_S(W, V')$ open such that

(i) $T(\mathcal{N}) \subseteq C^r(U,V)$ and
(ii) $T|_{\mathcal{N}} : \mathcal{N} \longrightarrow C^r_S(U,V)$ is continuous




we fix f on $U \setminus W$ We modify f on W.

v)

Proof: $\mathfrak{A} = (\varphi_i, U_i)$ let be a loc. finite collection of charts covering $Bd(W)$, the topological boundary of $W$ in $U$.

W.l.o.g. we can assume that every $\overline{U_i}$ is compact

1 We choose a shrinking $(L_i)_{i \in I}$.

We define $N \subseteq C^r_S(W, V')$ as follows

$$N = \{ h : W \to V' \mid \forall_{i \in I} \forall y \in \varphi_i(\overline{L_i}) \cap W$$

$$\| D^k(h\varphi_i^{-1})(y) - D^k(f\varphi_i^{-1})(y) \|_k$$

$$< d(y, \varphi_i(U_i \setminus W))^{1+\frac{1}{2}}$$

Then $N$ is open in $C^r_S(W, V')$

and $T(N) \subseteq C^r(U, V)$ and $T|_N$ is continuous. $\square$

End of Lecture 14.03.2023

## Proof of Theorem 2.17:

Uniqueness: Let $\beta$ and $\gamma$ be $C^s$-diff structures $\subseteq \alpha_r$.
Then
$$\mathrm{Diff}^r((M,\beta),(M,\gamma)) = \mathrm{Diff}^r((M,\alpha_r),(M,\alpha_r)) \ni id_M$$

Thm 2.14 $\Rightarrow$ $\mathrm{Diff}^r(M,M)$ is open in $C^r_S(M,M)$
Thm 2.16 $\Rightarrow$ $C^s(M,\beta,M,\gamma)$ is dense in $C^r(M,\beta,M,\gamma)$
$\Rightarrow$
$$C^s(M,\beta,M,\gamma) \cap \mathrm{Diff}^r(M,M)$$
is non-empty.

$\Rightarrow$ $\mathrm{Diff}^s(M,\beta,M,\gamma) \neq \emptyset$.

Existence: Zorn's Lemma

$\Rightarrow \exists\, B \subseteq M$ open: $\exists\, \beta$ $C^s$-diff structure on $B$ s.t. $\beta \subseteq \alpha_r|_B$.

any such $\mathrm{Max}(B,\beta)$ is maximal with those properties.

Assume $\mathcal{B} \neq \mathcal{U}$.

$\Rightarrow (\varphi, U) \in \mathcal{A}_r$ : $U \cap (\mathcal{U} \setminus \mathcal{B}) \neq \emptyset$

If $U \cap \mathcal{B} = \emptyset$ then

$\mathcal{B} \cup \{(\varphi, U)\}$ is a $C^r$-atlas on

$\mathcal{B} \cup \mathcal{U} \supsetneq \mathcal{B}$ $\lightning$

So $\underbrace{U \cap \mathcal{B}}_{=: W} \neq \emptyset$.

Then $W \subseteq U$ open and by Lemma 2.19

$\exists$ $\mathcal{N} \subseteq C^r_S(W, \varphi(W))$ open

$T : \mathcal{N} \longrightarrow C^r_S(U, \varphi(U))$

given in Lemma 2.19 (with $f = \varphi$)
is continuous.


$$T(\psi_0) = \begin{cases} \psi_0 , & W \\ \varphi , & U \setminus W \end{cases}$$

We can choose $\mathcal{N} \subseteq \mathrm{Diff}^r(U, \varphi(U))$ by 2.19.

Thm 2.16 $\Rightarrow N \cap C^s_S((W, \beta|_W), \varphi(W)) \neq \emptyset$

Take $\psi_0 \in$

We could have taken $N$ smaller such that

$$T(N) \subseteq \mathrm{Diff}^r(W, \varphi(W))$$

because $T$ is continuous and $\mathrm{Diff}^r(W, \varphi(W))$ is open in $C^r_S(W, \varphi(W))$.

Then
- $T(\psi_0) \in \mathrm{Diff}^r(W, \varphi(W))$
- $T(\psi_0)|_W \in \mathrm{Diff}^s((W, \beta), \varphi(W))$

Then $\beta \cup \{(T(\psi_0), U)\}$ is a $C^s$-atlas for $B \cup U$. ↯ $\square$

<u>Remark 2.20</u> Let us give the strategy of the proof of Thm 2.17 by an example.

$M = \mathbb{R}$ with $C^1$-structure given by

$$\{(\mathrm{id}_{]-1,\infty[} = \varphi, ]-1,\infty[),$$

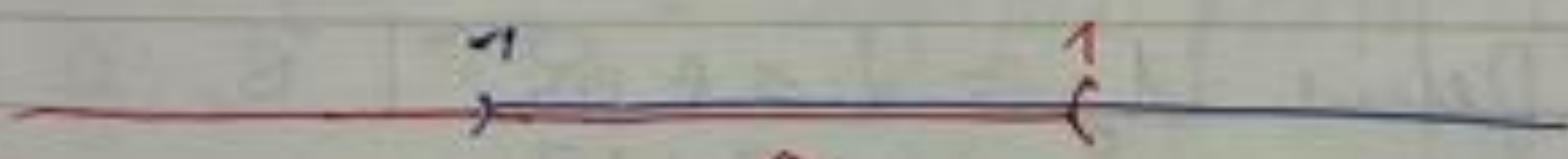
$$(f, ]-\infty, 1[)\}$$

with $f(x) = \begin{cases} \left(x+\frac{1}{2}\right)^2 - \frac{1}{4}, & 0 \le x < 1 \\ \frac{1}{4} - \left(-x+\frac{1}{2}\right)^2, & x < 0 \end{cases}$

It is not a $C^2$- atlas, because the overlap of $\varphi$ with $f$ is not $C^2$.

How to get a $C^2$- atlas?

- $(\varphi, ]-1, \infty[)$ gives a $C^2$-structure on $]-1, \infty[$.




Change f here.

- We change the chart $(f, ]-\infty, 1[)$ on $]-1, 1[$.

$$\tilde{f}(x) := \begin{cases} \frac{1}{4} - \left(-x+\frac{1}{2}\right)^2, & -\infty < x \le 0 \\ \frac{2}{3}x^3 - x^2 + x, & 0 \le x < 1 \end{cases}$$

Now $\{(\varphi, ]-1, \infty[), (\tilde{f}, ]-\infty, 1[)\}$

is a $C^2$— atlas.

We now come to the density theorem Thm. 2.16.

<u>Def 2.21</u>: (convolution)

(a) Let $U \subseteq \mathbb{R}^m$ open and $\sigma > 0$ such that $U$ contains a closed ball of radius $\sigma$.
Let $\theta \in C^\infty(\mathbb{R}^m, \mathbb{R}^{\geq 0})$ such that $\mathrm{supp}(\theta) \subseteq \overline{B}_\sigma(\underline{0})$.

Define $U_\sigma := \{ \underline{x} \in \mathbb{R}^m \mid \overline{B}_\sigma(\underline{x}) \subseteq U \}$

We define a map $C^0(U, \mathbb{R}^n) \to C^\infty(U_\sigma, \mathbb{R}^n)$
$f \longmapsto \theta * f$ via

$$(\theta * f)(\underline{x}) := \int_{\overline{B}_\sigma(\underline{x})} \theta(\underline{x} - \underline{y})\, f(\underline{y})\, d\underline{y}$$

The map $\theta * f$ is called convolution of $\theta$ with $f$.

(b) A map $\theta \in C^\infty(\mathbb{R}^m, \mathbb{R}^{\geq 0})$ with support in $\overline{B}_\sigma(0)$, $\sigma$ finite, is called convolution kernel if

$$\int_{\mathbb{R}^m} \theta(\underline{x})\, d\underline{x} = 1$$

(c) (a notation) Let $f \in C^r(U, \mathbb{R}^m)$ and $K \subseteq U$ be compact. We put

$$\|f\|_{r,K} := \sup\{\|D^{(k)} f(\underline{x})\|^{(k)} \mid \underline{x} \in K,\ k = 0, \dots, r\}$$

Remark 2.22: (i) $D^{(k)}(\theta * f) = (D^{(k)}(\theta)) * f$

(ii) Suppose $f \in C^{(k)}(U, \mathbb{R}^m)$, then

$$D^{(k)}(\theta * f) = \theta * D^{(k)}(f)),$$

because, by substituting $\underline{z} := \underline{x} - \underline{y}$, we obtain

$$(\theta * f)(\underline{x}) = \int_{\bar{B}_\sigma(\underline{0})} \theta(\underline{z})\, f(\underline{x}-\underline{z})\, d\underline{z}.$$

(iii) (approximation)

Let $U \subseteq \mathbb{R}^m$ be open, non-empty, $K \subseteq U$ be compact and $f \in C^r(U, \mathbb{R}^m)$, $0 \leq r \leq \infty$.

Let $\varepsilon > 0$. Then $\exists\, \sigma > 0$:
(a) $K \subseteq U_\sigma$ and
(e) $\forall_\theta$, convolution kernel with $\operatorname{supp}\theta \subseteq \bar{B}_\sigma(0)$

we have

$$\|\theta * f - f\|_{r,K} < \varepsilon.$$

Proof:- By (ii) we have $D^{(k)}(\theta * f)$
$= \theta * (D^{(k)}(f))$, so we can show (iii) for $D^{(k)}(f)$ separately, $k = 0, \dots, r$, seeing them as continuous maps. Thus we can w.l.o.g restrict to $r = 0$.

Take an open set $W \subseteq \mathbb{R}^m$ such that

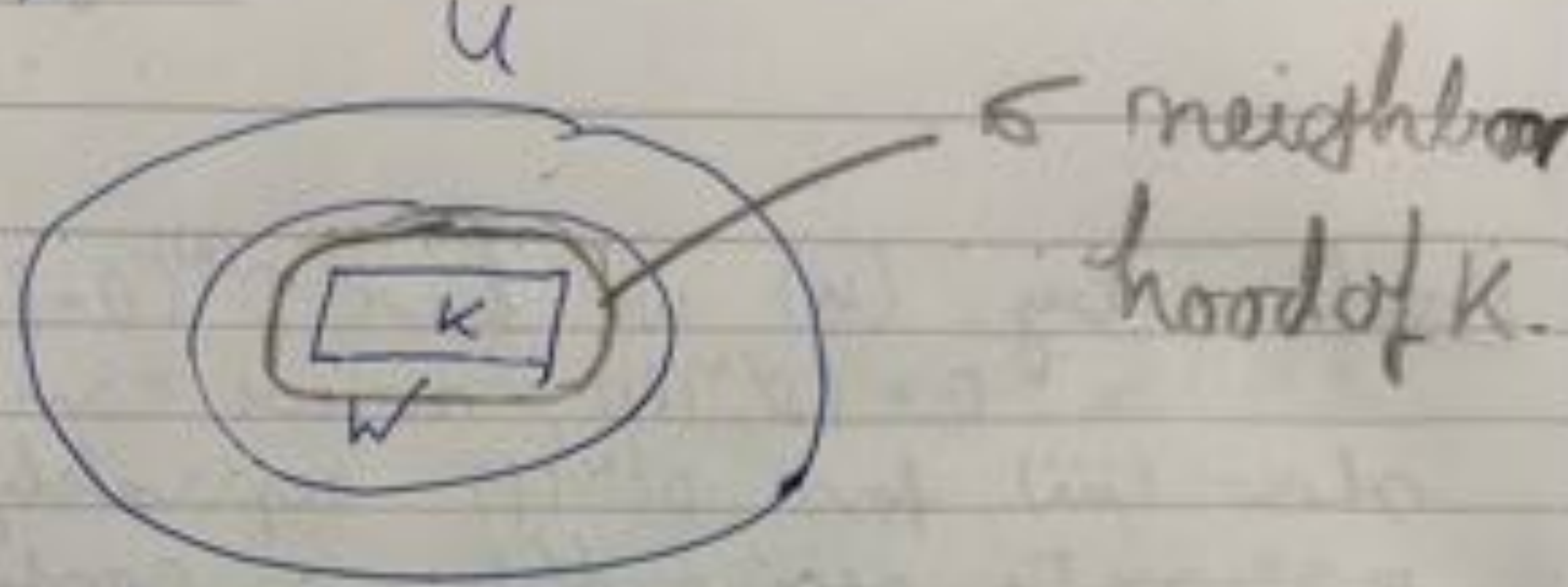
$$K \subseteq W \subseteq \overline{W} \subseteq U.$$

and such that $\overline{W}$ is compact.
(One can take $W$ to be a union of finitely many ~~tot~~ open balls.)

$f$ is continuous on $\overline{W}$ and $\overline{W}$ is compact $\Longrightarrow$ $f$ is uniformly continuous on $\overline{W}$.
Thus $\exists \quad \sigma > 0 : \forall \underline{x}, \underline{y} \in \overline{W}$:

$$( d_2(\underline{x}, \underline{y}) \leq \sigma \Rightarrow d_2(f(\underline{x}), f(\underline{y})) < \varepsilon )$$

Take the $\sigma$ to be smaller than $d_2(K, \mathbb{R}^m \setminus W)$.




Then $|\theta * f(\underline{x}) - f(\underline{x})|_2$

$$= \left| \int_{\bar{B}_\delta(\underline{0})} \theta(\underline{y})\left(f(\underline{x}-\underline{y}) - f(\underline{x})\right) d\underline{y} \right|_2$$

$$\underset{\substack{\uparrow \\ \theta(\underline{y}) \geq 0}}{\leq} \int_{\bar{B}_\delta(\underline{0})} \theta(\underline{y}) \left| f(\underline{x}-\underline{y}) - f(\underline{x}) \right|_2 d\underline{y}$$

$$\leq \varepsilon \int_{\bar{B}_\delta(\underline{0})} \theta(\underline{y})\, d\underline{y} = \varepsilon$$

for $\underline{x} \in K$ $\square$

End of Lecture 17.03.2023